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Mass Formula for Kerr Black Holes

Larry Smarr*

Center for Relativity Theory, Physics Department, The University of Texas at Austin, Austin, Texas 78712

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A new mass formula for Kerr black holes is deduced, and is constrasted to the mass formula which is obtained by integrating term by term the mass differential and which consists of three terms interpreted, respectively, as the surface energy, rotational energy, and electromagnetic energy of the charged rotating black hole. A comparison is suggested between a rotating black hole and a rotating liquid drop which leads to a speculation that Kerr black holes may develop instabilities.

The three-parameter, charged, Kerr solution[1] of the Einstein-Maxwell equation represents a charged, rotating black hole. These parameters may be taken to be the total mass M, the specific angular momentum $a = L/M$, and the charge Q of the black hole, with the range

$$0 \leqslant a^2 + Q^2 \leqslant M^2.$$

The surface of the black hole is the two-dimensional surface formed by the intersection of the event horizon with a spacelike hypersurface. As Hawking[2] points out, the surface area of a black hole can never decrease. For a charged Kerr black hole the area is constant, given by the expression

$$A = 4\pi[2M^2 + 2(M^4 - L^2 - M^2Q^2)^{1/2} - Q^2].$$

On inverting this relation, one obtains the mass[3] as a function of area A, angular momentum L, and charge Q:

$$M = \left[\frac{A}{16\pi} + \frac{4\pi L^2}{A} + \frac{Q^2}{2} + \frac{\pi Q^4}{A}\right]^{1/2}.$$

Let us use the mass differential dM to define three physical invariants of the black hole horizon:

$$dM = T\,dA + \Omega\,dL + \Phi\,dQ,$$

where, following Christodoulou[4] and Bekenstein,[5] we say T = effective surface tension, Ω = angular velocity, and Φ = electromagnetic potential.

The main purpose of this Letter is to point out that *the mass can be expressed in terms of these same quantities as a simple bilinear form*

$$M = 2TA + 2\Omega L + \Phi Q.$$

This striking formula follows by applying Euler's theorem on homogeneous functions to M, which is homogeneous of degree $\frac{1}{2}$ in (A, L, Q^2).

Remarkably, T, $\boldsymbol{\Omega}$, and Φ can be defined and are constant on the horizon for any stationary, axisymmetric black hole.[6] Moreover, the differential and integral mass formulas given above are formally similar to those derived for relativistic stars from a variational principle.[7] Thus, one would expect that the result reported above can be extended from the charged Kerr black hole to more general situations with matter present. In fact, this expectation has been confirmed by Bardeen, Carter, and Hawking.[8]

It should be understood that the above mass formula does *not* result from integrating the mass differential dM term by term. It is to this method for obtaining M that we now turn.

Since dM is a perfect differential, one is free to choose any convenient path of integration in (A, L, Q) space. In particular, one can choose a path which will define for a charged Kerr black hole three energy components: the surface energy E_s by

$$E_s = \int_0^A T(A', 0, 0)\,dA';$$

the rotation energy E_r by

$$E_r = \int_0^L \Omega(A, L', 0)\,dL', \quad A \text{ fixed};$$

and the electromagnetic energy E_{em} by

$$E_{em} = \int_0^Q \Phi(A, L, Q')\,dQ', \quad A, L \text{ fixed.}$$

These integrals may be directly evaluated using the variational definitions

$$T = \frac{1}{M}\left[\frac{1}{32\pi} - \frac{2\pi L^2}{A^2} - \frac{\pi Q^4}{2A^2}\right],$$

$$\Omega = \frac{4\pi L}{MA}, \quad \Phi = \frac{1}{M}\left[\frac{Q}{2} + \frac{2\pi Q^3}{A}\right].$$

The result is most easily expressed in terms of a new parameter set[9] (η, β, ϵ) related to the set (A, L, Q) by

$$\eta = (A/4\pi)^{1/2}, \quad \beta = a/\eta, \quad \epsilon = Q/\eta.$$

The integrated mass formula is then given by

$$M = \tfrac{1}{2}\eta(1+\epsilon^2)(1-\beta^2)^{-1/2},$$

$$E_s = \tfrac{1}{2}\eta,$$

$$E_r = \tfrac{1}{2}\eta[(1-\beta^2)^{-1/2} - 1],$$

$$E_{em} = \tfrac{1}{2}\eta\epsilon^2(1-\beta^2)^{-1/2},$$

with

$$M = E_s + E_r + E_{em}.$$

Christodoulou[3] already has shown by a different approach that for an *uncharged* Kerr black hole the mass decomposes into an irreducible mass $M_{ir} = \frac{1}{2}\eta$ and a rotational energy $M - M_{ir}$. Here the decomposition is extended to the charged case. It is seen also that if T is interpreted as surface tension, then M_{ir} should be interpreted as the *surface energy* of a black hole.

It is of interest that if one defines the moment of inertia I of a black hole by $I = L/\Omega$, then in the limit of small rotation ($\beta \to 0$) and small charge ($\epsilon \to 0$)

$$E_r \cong \tfrac{1}{2}I\Omega^2,$$

$$E_{em} \cong \tfrac{1}{2}Q^2\eta^{-1} + \tfrac{1}{4}Q^2\Omega^2\eta.$$

The interpretation of T as surface tension and Ω as angular velocity suggests a comparison of general relativistic rotating black holes with Newtonian rotating liquid drops.[10] By holding (1) the area of the Kerr black hole fixed, and (2) the volume, density, and surface tension of the liquid drop fixed, while increasing the angular momentum, one may compare the two by their one-parameter stationary equilibrium configurations. The qualitative behavior of the angular velocity, angular momentum, and ratio of surface energy to rotation energy for the two sequences is quite similar.[11]

Furthermore, the Gaussian curvature[9] of the surface of a Kerr black hole develops in a remarkably similar way to that of a rotating liquid drop. In both cases, as the angular momentum increases, the surfaces become flatter at the poles until zero Gaussian curvature occurs. Thereafter, the Kerr black holes acquire polar caps of negative curvature, while the rotating liquid drops acquire annuli of negative curvature centered on the poles. The endpoint of each sequence occurs when a topology change takes place for the surface.

Now in the liquid drop sequence, both secular and dynamic instabilities occur before the Gaussian curvature becomes zero on the poles.[10] The similarities discussed above, although giving only a qualitative analogy, suggest a speculation that instabilities may likewise develop for Kerr black holes when the ratio of angular momentum to mass is close to but less than that value where zero Gaussian curvature appears,[9] namely

$$L/M^2 = \tfrac{1}{2}\sqrt{3} \cong 0.866.$$

Work currently in progress by W. Press and S. Teukolsky and by J. R. Ipser should settle this question definitely.

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PHONON SPECTRA OF TETRAHEDRALLY BONDED SOLIDS. D. Weaire and R. Alben [Phys. Rev. Lett. 29, 1505 (1972)].

Equation (1) should read

$$V=\tfrac{3}{4}\alpha\sum_{l\Delta}[(\vec{u}_l-\vec{u}_{l\Delta})\cdot\hat{r}_\Delta(l)]^2+\tfrac{3}{16}\beta\sum_{l\{\Delta\Delta'\}}[(\vec{u}_l-\vec{u}_{l\Delta})\cdot\hat{r}_{\Delta'}(l)+(\vec{u}_l-\vec{u}_{l\Delta'})\cdot\hat{r}_\Delta(l)]^2,$$

and Eq. (3) should read

$$(m\omega^2-4\alpha)a_\Delta(l)=-\alpha\Big[\sum_{\Delta'\neq\Delta}a_{\Delta'}(l\Delta')-3a_\Delta(l\Delta)\Big].$$