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Classical, n-Component Spin Systems or Fields with Negative Even Integral n

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An n-component classical spin system with arbitrary pair interactions, or a lattice field theory, is considered, in arbitrary dimensionality d, with spin weighting factor $\exp[-w(s^2)]$, or equivalently local interactions satisfying $w(y)=O(y^{m+\delta})$ as $y\to 0$ with $m=1,2,\ldots$ and $\delta>0$. Analytic continuation in n is defined and used to show that for $n=-2k$ with $k=1,2,\ldots,m$ the zero-field free energy and all isotropic correlation functions and cumulants take their Gaussian (or free field) values.

The critical behavior of a classical continuous spin system or, equivalently, a lattice field theory, with n-component variables $\vec{s}_i=(s_i^{\alpha})=(s_i^{(1)}, s_i^{(2)},\ldots,s_i^{(n)})$ has been studied as a function of the spatial or lattice dimensionality d in the limits (a) $d\to 4-$ for short-range forces,[1-4] and for short-range plus dipolar forces[5]; (b) $d\to 2\sigma-$ for long-range interactions decaying as $1/r^{d+\sigma}$ [6-8]; and[9] (c) $n\to\infty$ for both short-range,[3,8,10] and long-range interactions.[6,8,11] In addition the limit $n\to 0$ has been shown[12] to be applicable to the self-avoiding walk problem.

Recently, Balian and Toulouse[13] considered the case $n=-2$. By examining the diagrammatic expansion term by term for a single-spin (or self-) interaction $w=u_0|\vec{s}|^4$, they developed a continuation in n and showed (d) that for $n=-2$, in zero field, only the quadratic or free-field terms in the Hamiltonian play a role. Thus the free energy, and hence the exponent α take on their Gaussian values for all d when $n=-2$. The diagrams for the two-point correlation function $G_2=\langle s_i^{\alpha}s_j^{\alpha}\rangle$ were likewise shown to lead to Gaussian behavior, whence the exponents γ, ν, and η take their Ornstein-Zernike values, for all d. Balian and Toulouse indicated that these conclusions should hold also for interactions of higher order than $u|\vec{s}|^4$.

In this note we first show how to effect an explicit analytic continuation in the number of components, n, without recourse to a diagrammatic expansion. This reveals rather clearly why $n=-2$ plays a special role, and provides a more transparent derivation of the Balian-Toulouse results. In addition our treatment shows that not only the two-point function but also *all* the higher-order, isotropic l-point correlation or cumulant functions in zero field exhibit Gaussian or free-field behavior. Finally the formulation yields these same results for $n=-2m$, $m=1,2,\ldots$ for systems in which the single-spin interaction $w(s^2)$ satisfies

$$w(y)=O(y^{m+\delta}) \text{ as } y\to 0, \tag{1}$$

with $\delta>0$. Thus, for example, an interaction of the form $v|\vec{s}|^6$ exhibits Gaussian behavior for both $n=-2$ and $n=-4$. Furthermore $w(y)$ need not be a polynomial, so that the interaction $t|\vec{s}|^{5/2}$ also yields Gaussian behavior for $n=-2$.

We consider explicitly a system of N spins with reduced Hamiltonian

$$\bar{\mathcal{H}}\{\vec{s}\}=\frac{-\mathcal{H}}{k_BT}=\tfrac{1}{2}\sum_{i\neq j}\sum K_{ij}\vec{s}_i\cdot\vec{s}_j, \tag{2}$$

where $K_{ij}=K_{ji}$. For each spin there is a weighting factor

$$f_i = f_i(\vec{s}_i) = \exp[-\kappa_i s_i^2 - w_i(s_i^2)], \quad s_i^2 = \sum_{\alpha=1}^{n} (s_i^\alpha)^2, \tag{3}$$

which is assumed to be integrable on $-\infty < s_i^\alpha < \infty$ for all $\kappa_i > 0$. The Gaussian or free-field model corresponds to $w_i \equiv 0$. The partition function is then

$$Z_N^{(n)}\{\kappa, w\} = \prod_{k=1}^{N} \int d^n s_k\, f_k(\vec{s}_k) \exp[\mathcal{H}\{\vec{s}\}]. \tag{4}$$

In order to see how a continuation to negative n may be effected,[14] it is instructive to consider the single-spin partition function

$$Z_1^{(n)}\{\kappa, w\} = C_n \int_0^\infty s^{n-1} \exp[-\kappa s^2 - w(s^2)]\, ds, \tag{5}$$

where $C_n = 2\pi^{n/2}/\Gamma(\frac{1}{2}n)$. To this end we suppose generally that

$$[\![e^{-w(y)}]\!]_{k+1} \equiv e^{-w(y)} - \sum_{l=0}^{k} a_l(-y)^l = O(y^{k+\delta}), \tag{6}$$

as $y \to 0$. Note that when (1) holds we have

$$a_0 = 1, \quad a_1 = a_2 = \cdots = a_m = 0. \tag{7}$$

Using the expression (6) in (5) yields, for $\kappa > 0$,

$$Z_1^{(n)}\{\kappa, w\} = C_n \int_0^\infty s^{n-1} \exp(-\kappa s^2) [\![\exp[-w(s^2)]]\!]_{k+1}\, ds + (\pi/\kappa)^{n/2} \sum_{l=0}^{k} (-)^l a_l \Gamma(\tfrac{1}{2}n + l)/\Gamma(\tfrac{1}{2}n)\kappa^l. \tag{8}$$

The subtracted integral now converges down to $n = -2k$ where, however, it makes no contribution since $C_{-2k} = 0$ $(k = 0, 1, \ldots)$. The sum of terms may likewise be continued analytically in n. If we take $k = m$, only the first term survives when (1) and (8) hold, yielding

$$Z_1^{(-2m)}\{\kappa, w\} = \kappa^m/\pi^m = Z_1^{(-2m)}(\kappa, 0). \tag{9}$$

This is evidently independent of the properties of w and equal to the Gaussian value ($w \equiv 0$).

This analysis of $Z_1^{(n)}$ is rigorous; a similar but more tedious calculation, utilizing an expansion in powers of K_{12}, verifies the same conclusion for $N = 2$ under equivalent conditions provided the determinant

$$D_N\{\kappa\} = \det(\underline{K}) = |\kappa_i \delta_{ij} - \tfrac{1}{2}K_{ij}| \tag{10}$$

remains positive. For general N, however, we present a more formal but rapid method (not unlike that used in Ref. 9). We suppose that expansions for w_i like (6), with coefficients a_{il}, may be extended to $k = \infty$ [as certainly true if $w_i(y)$ is a polynomial]. We then write (3) and (4) as

$$Z_N^{(n)}\{\kappa, w\} = \prod_{k,k'=1}^{N} \left[\int d^n s_{k'} \sum_{l=0}^{\infty} a_{kl} (\partial/\partial \kappa_k)^l \right] \exp[-\sum_{i,j} (\kappa_i \delta_{ij} - \tfrac{1}{2}K_{ij}) \vec{s}_i \cdot \vec{s}_j], \tag{11}$$

where the differential operators serve to regenerate the full weighting factors. On interchanging the order of summation and integration, the integrals factorize into n Gaussian integrals on the s_i^α, yielding

$$Z_N^{(n)}\{\kappa, w\} = \prod_{k=1}^{N} \left[\sum_{l=0}^{\infty} a_{kl} \left(\frac{\partial}{\partial \kappa_k} \right)^l \right] \left[\frac{\pi^N}{D_N\{\kappa\}} \right]^{n/2}. \tag{12}$$

At this point the analytic continuation in n is easily made. While the determinant D_N does not vanish (as for T exceeding the Gaussian critical point T_0), there is no ambiguity. Now suppose $n = -2$: the determinant D_N then appears *linearly* in (12), and by (10), so does each κ_k. Hence the only effective terms in the differential operators are those with coefficients $a_{k0} = 1$ and $a_{k1} = 0$ (since $m \geq 1$). The reason for the special role played by $n = -2$ is thus obvious.

More generally, for $n = -2m$ the mth power of the determinant yields a polynomial of order m in each κ_k, so that only the coefficients a_{kl} for $l \leq m$ can play a role; but in view of (7) these all vanish for $l > 0$. Consequently, we finally obtain

$$Z_N^{(-2m)}(\kappa, w) = \pi^{-mN} |\kappa_i \delta_{ij} - \tfrac{1}{2}K_{ij}|^m, \tag{13}$$

which is just the Gaussian result for $n = -2m$.

This establishes our conclusions for the free energy. In order to derive the correlation functions and the cumulants we simply note that

$$G_2{}^0 = \langle \vec{s}_k \cdot \vec{s}_l \rangle = n\langle s_k{}^\alpha s_l{}^\alpha \rangle = \frac{\partial \ln Z_N{}^{(n)}\{K_{ij}\}}{\partial K_{kl}}, \qquad (14)$$

$$G_4{}^0 = \langle \vec{s}_k \cdot \vec{s}_l \vec{s}_{k'} \cdot \vec{s}_{l'} \rangle_{\text{cum}} = \frac{\partial^2 \ln Z_N{}^{(n)}\{K_{ij}\}}{\partial K_{kl} \partial K_{k'l'}}, \qquad (15)$$

and so on for the higher-order isotropic cumulants. Since, even as a functional of the K_{ij}, the partition function (13) takes the Gaussian value when $n = -2m$, all the isotropic correlation functions likewise remain identical to their Gaussian analogs. We note, however, that an *anistropic* correlation function, such as $\langle s_k{}^\alpha s_l{}^\alpha s_{k'}{}^\beta s_{l'}{}^\beta \rangle$, cannot be evaluated in this way and is, presumably, *not* equal to its Gaussian analog (even in zero field).

To justify the interchange of limits involved in going from (11) to (12), attention must be given to the behavior of $w(y)$ for large y. If $w(y)$ is a polynomial, it is sufficient to multiply it by the cutoff factor $e^{-\mu y}$ and then let $\mu \to 0+$, after continuing in n and performing the summations on l.

It is interesting to note that our evaluation of $Z_N{}^{(n)}$ holds for $n = 0$, where it simply yields the Gaussian result, namely, unity. For the excluded volume problem, however, we presumably require $\lim(n \to 0)[Z_N{}^{(n)}]^{1/n}$. This brings in $\ln D_N\{\kappa\}$ and its powers, as the operand for the differential operators. This also happens when one tries to develop expansions about $n = -2m$.

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Laboratory Observation of Slow-Mode Shock Waves

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We have identified a slow-mode shock wave in a laboratory experiment. We also present a model of the novel method of launching the shock.

There have been many reports of laboratory experiments which produce fast-mode shock waves propagating into plasmas containing magnetic fields of various magnitudes and directions.[1-3] Such shocks are associated with the steepening of fast-mode magnetohydrodynamic (MHD) waves. Slow-mode MHD waves can also, in principle, steepen into shock waves.[4] Space observations[5,6] have revealed discontinuities in the solar wind, which satisfy the conservation relations for such slow shocks. We present here what we believe to be the first laboratory observation of slow-mode shock waves.

The main differences between fast and slow shocks are described by Kantrowitz and Petschek.[4] Density increases across both, but the transverse component of magnetic field increases across fast shocks and decreases across slow shocks. Thus in slow shocks both magnetic and kinetic energy are converted into thermal energy, and it is the thermal pressure which drives the shock against the magnetic back pressure.