sudden decrease in intensity interpreted as the intersection of the spin-wave and Stoner modes is temperature independent, although measurements at the higher temperatures are difficult since the spin-wave intensity is reduced at all energies. It is surprising that the intersection point does not decrease with temperature since one would expect the Stoner continuum to drop in energy as the band splitting decreases. We believe that our interpretation of the sudden decrease of spin-wave intensity as being due to Stoner modes is reasonable especially in view of the band calculations of Cooke[8,9] and Mook *et al.*[1] Certainly the falloff is intensity is not an artifact of the spectrometer since many measurements have been made to energies much higher than 90 meV in other materials, and in the [100] direction in nickel spin waves are easily measured past 100 meV. It is to be noted that even though the spin-wave dispersion curves are isotropic the Stoner continuum may be quite anisotropic. The band calculations clearly show this effect and in fact predict that the spin-wave intensity should fall off more rapidly in the [111] direction than in the [100] direction.[9] The existence of spin waves at high temperatures is perhaps reasonable if long-range spin correlations persist at elevated temperatures. It is not clear if the spin-wave Stoner band intersection can also be explained in these terms.

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## Dipolar Interactions at Ferromagnetic Critical Points

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The critical behavior for $T > T_c$ and $H = 0$ of ferromagnets with both isotropic exchange coupling and dipolar interactions is studied by exact renormalization-group techniques in $d = 4 - \epsilon$ dimensions ($\epsilon > 0$) with ($n = d$)-component spins. A crossover from short-range isotropic (Heisenberg) to characteristic dipolar behavior occurs. The asymptotic spin-spin correlation function develops a factor $\delta_{\alpha\beta} - q^\alpha q^\beta/q^2$ which suppresses longitudinal fluctuations. Experiments on EuO and EuS are considered.

Magnetic dipole-dipole interactions exist in all magnetic materials but in most ferromagnets the ordering is caused by exchange coupling, which, in materials with $T_c > 300°\mathrm{K}$, strongly dominates the dipolar terms. We suppose the magnetic Hamiltonian can be written in the localized spin form

$$\mathcal{H} = -\tfrac{1}{2}\sum_{\vec{R},\vec{R}'}\{J(\vec{R}-\vec{R}')\vec{S}_{\vec{R}}\cdot\vec{S}_{\vec{R}'} + (g_s\mu_B)^2\sum_{\alpha,\beta}\mathcal{A}^{\alpha\beta}(\vec{R}-\vec{R}')S_{\vec{R}}{}^{\alpha}S_{\vec{R}'}{}^{\beta}\} \quad (1)$$

with, in $d$ dimensions, the dipolar coupling

$$\mathcal{A}^{\alpha\beta}(\vec{R}) = d(R^\alpha R^\beta)/R^{d+2} - \delta_{\alpha\beta}/R^d, \quad (2)$$

while $J(\vec{R})$ is a short-range coupling of strength

$$\hat{J} = \sum_{\vec{R}} J(\vec{R}). \quad (3)$$

Note that in (1) we take the number of spin components, $n$, equal to the dimensionality $d$, since the dipole-dipole interaction involves scalar products like $\vec{S}_{\vec{R}}\cdot(\vec{R}-\vec{R}')$ which couple spin and

lattice spaces. The relative magnitude of the dipolar terms can (for $d=3$) then be measured by

$$\hat{g}=4\pi g_s{}^2\mu_B{}^2/3v_a\hat{J}=\Delta T_{\rm dip}/T_c, \tag{4}$$

in which $v_a$ is the cell volume of the (cubic) lattice. This parameter corresponds simply to the fractional increase in the overall mean field critical temperature $T_0=T_1(1+\hat{g})$, relative to the exchange-only value $T_1=\frac{1}{3}S(S+1)\hat{J}=T_c/\theta$ where, for the nearest-neighbor fcc Heisenberg lattice with $S=\frac{1}{2}$ to $\infty$ one has[1] $\theta=T_c/T_1\simeq 0.68$ to 0.80. If, as customary, one puts $p=g_{\rm eff}=[S(S+1)]^{1/2}g_s$, one obtains the second part of (4) with $\Delta T_{\rm dip}=0.87\times[p^2\theta/v_a(\text{Å}^3)]$°K. For Ni, EuO, and EuS, with $T_c\simeq 627$, 69.3, and 16.3°K, respectively, one finds[2,3] $\Delta T_{\rm dip}\simeq 0.02$, 1.2, and 0.8°K. These figures confirm the opening remarks, and indicate that for Ni the experimentally observed critical behavior should be characteristic of short-range coupling only.[2] On the other hand, in view of the long-range and singular character of the dipolar interactions, it is reasonable to expect some new, characteristically dipolar behavior to appear for $t=(T-T_c)/T_c\lesssim t^{\times}=\hat{g}^{1/\phi}$, where $\varphi$ is some (as yet unknown) crossover exponent.[4] In particular, new values of the critical exponents $\alpha$, $\beta$, $\gamma$, $\nu$, $\eta$, etc.[1] should become evident. If, as we will show, $\phi\simeq 1.3$, one obtains $t^{\times}\simeq 4\times10^{-2}$ and $9\times10^{-2}$ for EuO and EuS, respectively. These estimates for the (reduced) crossover temperature are well within the ranges studied in recent magnetic,[5] neutron-scattering,[6] and specific-heat measurements.[7]

With the aid of Wilson's renormalization-group approach,[8] and the $\epsilon=4-d$ perturbation technique,[9,10] a serious attack on this problem is, for the first time, possible. In this note we report the results of calculations to leading order in $\epsilon$ using continuous-spin model with $n=d$ components, $s_{\vec{R}}{}^{\alpha}$, for each spin $\vec{s}_{\vec{R}}$, and $|\vec{s}|^2+|\vec{s}_{\vec{R}}|^4$ weighting factors.[8-11]

The main computation, to order $\epsilon$, uses the exact renormalization-group equations[9,11,12] for a reduced Hamiltonian of the form

$$\mathcal{H}=-\tfrac{1}{2}\int_{\vec{q}}\sum_{\alpha\beta}u_2{}^{\alpha\beta}(\vec{q})s_{\vec{q}}{}^{\alpha}s_{-\vec{q}}{}^{\beta}-\sum_{\alpha\beta}(u+v\delta_{\alpha\beta})\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''}s_{\vec{q}}{}^{\alpha}s_{\vec{q}'}{}^{\alpha}s_{\vec{q}''}{}^{\beta}s_{-\vec{q}-\vec{q}'-\vec{q}''}{}^{\beta}, \tag{5}$$

where $\int_{\vec{q}}$ means $(2\pi)^{-d}\int d^dq$, with the effective pair potential

$$u_2{}^{\alpha\beta}(\vec{q})=[r+q^2+f(q^{\alpha})^2]\delta_{\alpha\beta}+gq^{\alpha}q^{\beta}/q^2-hq^{\alpha}q^{\beta}. \tag{6}$$

As usual[9-11] $r$ is proportional to $T-T_0$, while the coefficient of $q^2$ is fixed at unity by a spin rescaling. The coefficients $g$, $h$, and $f$ arise directly from the Fourier transform of the dipolar interaction (2) and are all proportional to $\hat{g}$. The latter two terms are commonly neglected in discussions of dipolar interactions.[13] The $f$ term, although initially very small, has only the cubic symmetry of the underlying lattice, and via the renormalization procedure, generates the terms with coefficient $v$ in (5). Ultimately, i.e., very close to $T_c$, these terms may play an important role (which has not been fully elucidated); detailed estimation, however, indicates that their effects are unlikely to be seen with present experimental techniques.[14] Accordingly we may set $f=v=0$ in discussing the main effects of the dipolar interactions.

The Gaussian propagator for the graphical expansion then has the form

$$G^{\alpha\beta}(\vec{q})=\frac{\delta_{\alpha\beta}-q^{\alpha}q^{\beta}/q^2}{r+q^2}+\frac{q^{\alpha}q^{\beta}/q^2}{r+g+(1-h)q^2}. \tag{7}$$

For $g,h\ll r$, or equivalently, $t\gg\hat{g}$, this reduces to the standard propagator for the short-range, $(4-\epsilon)$-component, classical, isotropic (Heisenberg) spin model. We find the usual nontrivial fixed point with exponents[15,10]

$$1/\gamma\approx 1-\tfrac{1}{4}\epsilon,\quad \eta\approx 3\epsilon^2/12^2,\quad \alpha_s\approx-\tfrac{1}{8}\epsilon^2, \tag{8}$$

and $g=h=0$. This fixed point is strongly unstable with respect to the dipolar term $gq^{\alpha}q^{\beta}/q^2$; the corresponding crossover exponent[4,14] is found to be

$$1/\phi=1-\tfrac{1}{4}\epsilon+O(\epsilon^2), \tag{9}$$

confirming the value $\phi\simeq 1.3$ quoted above for $d=3$ ($\epsilon=1$).

Under iteration of the renormalization equations for $g>0$, one finds $g\to\infty$ leading to a new, dipolar, fixed point [with $h=O(1)$]. The corresponding characteristic dipolar exponents are

$$1/\gamma\approx 1-\tfrac{9}{34}\epsilon,\quad \eta\approx 20\epsilon^2/3\times17^2,$$
$$\alpha_s\approx-\tfrac{1}{34}\epsilon. \tag{10}$$

The value of $\eta$ quoted here follows[16] from a separate second-order Feynman-graph calculation,[10] which also confirms (11) to (13) below. The specific-heat exponent $\alpha_s$ is derived from the scal-

ing relation[1] $2-\alpha_s = d\nu \approx \frac{1}{2}d\gamma$ but has not otherwise been checked.

In the dipolar region $t \lesssim \hat{g}^{1/\phi}$, the true spin-spin correlation function has the form

$$\Gamma^{\alpha\beta}(\vec{q}) \approx C t^{-\gamma} \frac{\delta_{\alpha\beta} - q^\alpha q^\beta/q^2}{1+\xi^2 q^2}, \tag{11}$$

provided $0 < \xi^2 q^2 \ll 1$, where the correlation length varies as

$$\xi(T) \approx \xi_0 t^{-\nu}, \quad \nu = \gamma/(2-\eta), \tag{12}$$

and (10) applies. The factor $\delta_{\alpha\beta} - q^\alpha q^\beta/q^2$ in (11), which is absent in the short-range region $t > \hat{g}^{1/\phi}$, represents the suppression of longitudinal ($\vec{S}_{\vec{q}} \| \vec{q}$), low-momentum, spin fluctuations. It remains at the critical point itself where we have

$$\Gamma^{\alpha\beta}(\vec{q}) \sim \frac{\delta_{\alpha\beta} - q^\alpha q^\beta/q^2}{q^{2-\eta}}, \tag{13}$$

with $\eta$ given by (10). These results show that one could identify the "critical mode" with the transverse $|\vec{q}| \to 0$ fluctuations.

Finally for $t \ll \hat{g}^{1/\phi}$ the magnetostatic susceptibility $\chi^{\alpha\beta}$ takes the form expected from macroscopic theory, namely,

$$\frac{v_a k_B T}{g_s{}^2 \mu_B{}^2} \chi^{\alpha\beta} \approx \delta_{\alpha\beta}\left[C^{-1} t^{\gamma} + \left(\frac{g_s{}^2 \mu_B{}^2}{v_a k_B T}\right) D^\alpha\right]^{-1}, \tag{14}$$

where $C$ and $\gamma$ have the same values as in (11), while the "demagnetization factor" $D^\alpha$ is, for $d=3$, given by

$$D^\alpha = \tfrac{4}{3}\pi + v_a \textstyle\sum_{\vec{R}} [1 - 3\cos^2\theta^\alpha(\vec{R})]/R^3, \tag{15}$$

where $\theta^\alpha(\vec{R})$ denotes the angle between $\vec{R}$ and the $\alpha$ axis, while for the elliptical sample shape assumed, the lattice sum is independent of the origin site.[17] Note that (14) is *not* a simple $|\vec{q}| \to 0$ limit of (11).

From a theoretical viewpoint it is interesting that the value of $\eta$ is increased by the addition of dipolar interactions; by truncating (9) and (10) the change is estimated to be from about 0.021 to 0.023 for $d=3$. This is not in the direction (toward $\eta=0$) that might have been guessed[1] from the "long-range" character of the dipolar interactions. Similarly, the value of $\gamma$ is increased, from about 1.33 to 1.36 for $d=3$ (by truncating likewise), which is also away from the mean field result $\gamma=1$. This contrasts with the effects of long-range exchange interactions of the form $1/R^{d+\sigma}$ with $0 < \sigma < 2$.[18] Experimentally, one finds for nickel[2,19] $\gamma = 1.34 \pm 0.02$ (and similarly for iron). For EuO and EuS, neutron-scattering experiments[6] indicate $\gamma = 1.39 \pm 0.05$ which, despite the fairly large uncertainties, is clearly larger, as predicted. On the other hand, static measurements[5] on EuO suggest $\gamma = 1.29 \pm 0.03$. According to all current theoretical ideas the two techniques should yield the same exponent. However, it is quite possible that, in one or both experiments, the pure power-law region had not been reached; the apparent increase in $\gamma^*(T)$ in Fig. 3 of Ref. 5 for one acceptable choice of $T_c$, is suggestive from this viewpoint. In any case experiments with $t < 10^{-2}$ are clearly desirable to resolve this important discrepancy and, hopefully, to provide a test of the present theory.[20,21] It would also be interesting to observe the predicted angular dependence of the spin fluctuations by an appropriate technique.[22]

Comparison of (8) and (10) indicates that the dipolar specific-heat exponent should, in general, differ from that for short-range interactions. For three dimensions, evaluation of the leading terms alone would give $\alpha_s \simeq -0.03$ and $-0.13$, respectively; however, neglect of the (unknown) second-order terms in (10) is obviously unjustified in view of their relatively large magnitude in (8). Even so, the difference might correspond with the "nonuniversal" specific-heat exponents observed recently[7] for the ferromagnet EuO, namely $\alpha_s = -0.04 \pm 0.06$, and for the (Heisenberg-like) *anti*ferromagnet $RbMnF_3$, namely $\alpha_s = -0.14 \pm 0.04$, since, as might be anticipated, isotropic dipolar interactions do not affect the asymptotic behavior of *anti*ferromagnets (to order $\epsilon$),[16] which should, accordingly, display the short-range exponents. However, the close agreement with the truncated estimates is likely to be coincidental and, indeed, the significant rounding[7] of the EuO data for $t \lesssim 10^{-3}$ casts some doubt on the experimental interpretation. In any event the present theoretical analyses make it clear that the critical behavior of ferromagnets with low transition points, such as the europium chalcogenides, should no longer be expected to match that of otherwise comparable antiferromagnets.

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[13]See, e.g., A. I. Larkin and D. E. Khmel'nitskii, Zh. Eksp. Teor. Fiz. 56, 2087 (1969) [Sov. Phys. JETP 29, 1123 (1969)], who, in contrast to us, consider single-component (Ising-like) spins with dipolar interactions.

[14]Thus, the $v$ term represents a relevant operator for all the fixed points studied with $n=d=4-\epsilon$, and $\epsilon$ small, but the initial value is $v_0=0$ and the corresponding exponent $\lambda_v=d-\omega_v=1/\nu_v$ is quite small. Besides the Gaussian and the Heisenberg-like fixed points, there is also an Ising-like fixed point, with $u^*=0$, but this fixed point is unstable with respect to $u$. In both cases, $\lambda_f$ is of order $\epsilon^2$, and is negative so that $f$ is an irrelevant variable. See also Ref. 12; M. E. Fisher and P. Pfeuty, Phys. Rev. B 6, 1889 (1972); F. J. Wegner, Phys. Rev. B 6, 1891 (1972); A. Aharony, to be published.

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[21]Since the leading order dipolar and exchange-only coefficients for $\eta$ and $\gamma$ are relatively so close, the higher-order $\epsilon$ corrections could possibly alter the relative magnitudes for $d=3$.

[22]Neutron scattering with momentum transfer near $\vec{Q}=0$ reveals only the transverse fluctuations, but the scattering near other reciprocal lattice points is sensitive to the longitudinal fluctuations.

# Solvable Model of Stripping Reactions with Variable Spectroscopic Factor

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Model three-body calculations have been performed with two-channel separable interactions to test the predictions for angular distributions and spectroscopic factors by Butler, Hewitt, McKellar, and May, and distorted-wave–approximation theories of deuteron stripping. There are indications that both theories are inadequate.

One of the important parameters in nuclear-structure theory is the so-called "spectroscopic factor ($S$)"—the probability that an $(A+1)$-particle nucleus is simply an extra particle added to the ground state of an $A$-particle nucleus. It has long been recognized that this parameter can be determined from an analysis of stripping and pickup reactions,[1] but as one has no *a priori* knowledge of $S$, the value extracted from the reaction analysis is somewhat controversial.[2] We present the results of a model calculation in which $S$ is known and one can test the ability of the reaction theory to extract the correct value. We test both the distorted-wave Born-approximation (DWBA) and the Butler-Hewitt-McKellar-May (BHMM) theories.

Our model is an extension of the three-body separable potential model which has been applied