Chiral SU(2) ⊗ SU(2) Mixing and the Quark Model of Hadrons*

F. J. Gilman and M. Kugler†
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305
(Received 1 December 1972)

We present a simple form for the vector and axial-vector charges, transformed so that their matrix elements between (constituent) quark-model states correspond to measurable transitions between physical states. A comparison with experiment of predictions for pionic transitions is made.

The algebra of vector and axial-vector currents proposed by Gell-Mann[1] has, for some time now, been one of the accepted "truths" of hadron physics. Given the correctness of the algebra, one is immediately led to consider how the observed particle and resonance states transform under this algebra. From the Adler-Weisberger relation[2] itself, it is already clear that the observed hadron states at infinite momentum[3] do not fall into irreducible representations of the chiral SU(2) ⊗ SU(2) algebra of charges. The axial-vector charge connects the nucleon to many higher-mass N^* states which contribute to the sum rule and must then share the same irreducible representation of SU(2) ⊗ SU(2) with the nucleon. Correspondingly, the nucleon must have components in several irreducible representations of the algebra.

Although some progress and understanding have been achieved,[4] the problem of a complete classification of hadron states under the SU(2) ⊗ SU(2) charge algebra is unsolved. Furthermore, up to this point, much of the work on classifying the states has been on a case-by-case basis. For a systematic approach, one wants a transformation from the irreducible representations characteristic of the quark model to the reducible representations of the physical states.[5] In this paper, we assume such a transformation exists and choose to act with it on the charges rather than the states. Although the details of the transformation are unknown, we suggest that the transformed charges have a simple algebraic structure, allowing us to systematically relate many hadronic matrix elements.

We start by defining the chiral SU(2) ⊗ SU(2) algebra of charges at equal times,[6]

$$[Q^i, Q^j] = i\epsilon^{ijk} Q^k, \quad [Q_5{}^i, Q^j] = i\epsilon^{ijk} Q_5{}^k,$$
$$[Q_5{}^i, Q_5{}^j] = i\epsilon^{ijk} Q^k, \qquad (1)$$

where i, j, k run from 1 to 3 and Q and Q_5 are the space integrals of the time component of the vector and axial-vector currents, respectively. The operators $Q^i + Q_5{}^i$ and $Q^i - Q_5{}^i$ then form two commuting SU(2) algebras. Irreducible representations of hadrons moving at infinite momentum in the z direction are labeled as $\{(I_1, I_2)_{S_z}, L_z\}$, where I_1 and I_2 are the "isospin" under $Q^i + Q_5{}^i$ and $Q^i - Q_5{}^i$, respectively; S_z is the eigenvalue of $Q_5{}^0$, the singlet axial charge which corresponds to the intrinsic quark spin in a quark model of hadrons. The quantity L_z is defined then as $L_z = J_z - S_z$, J_z being the z component of the total angular momentum of the state. The isospin content of (I_1, I_2) ranges from $|I_1 - I_2|$ to $I_1 + I_2$.

Now consider the transformation V which takes one from the set of irreducible representations characteristic of quark constituents ($q\bar{q}$ for mesons, qqq for baryons) to the physical states which form complicated reducible representations of SU(2) ⊗ SU(2):

$$|\text{physical}\rangle = V|\text{constituents}\rangle.$$

The operator V, which contains the dynamics of the world, may be roughly thought of as, among other things, adding infinite numbers of $q\bar{q}$ pairs to the constituent $q\bar{q}$ or qqq state to form the physical meson or baryon. Then, assuming V is a unitary operator, the measurable matrix elements we wish to study are, for example,

$$\langle \text{physical}'|Q_5|\text{physical}\rangle,$$

which may be rewritten as

$$\langle \text{physical}'|Q_5|\text{physical}\rangle = \langle \text{constituents}'|V^{-1}Q_5V|\text{constituents}\rangle. \qquad (2)$$

The operator V and its properties have been studied in some detail in the free quark model by Melosh.[7] Most interestingly, he finds that while V, acting on a single irreducible representation, $|\text{constituents}\rangle$, generates a complicated infinite sequence of representations with increasing angular momentum and quark spin, the quantity $V^{-1}Q_5V$ is quite simple.[8] It transforms as a sum of the $\{(1, 0)_0, 0\}$-$\{(0, 1)_0, 0\}$ (*like* Q_5) and $\{(\frac{1}{2}, \frac{1}{2})_1, -1\}$-$\{(\frac{1}{2}, \frac{1}{2})_{-1}, 1\}$ representations of SU(2) ⊗ SU(2), respectively. We shall assume that the remarkable property of $V^{-1}Q_5V$ in the free quark model in terminating in only two terms is generally true. Furthermore, in order to have a very simple and elegant form, we will assume, in addition, that the $\{(1, 0)_0, 0\} - \{(0, 1)_0, 0\}$ operator is to be identified *entirely* with a multiple of Q_5, something which is not true in the free quark model. We thus write

$$V^{-1}Q^iV = Q^i, \tag{3a}$$

$$V^{-1}Q_5{}^iV = \cos\alpha\, Q_5{}^i + \sin\alpha K^i, \tag{3b}$$

where α is a constant and K transforms as $\{(\frac{1}{2}, \frac{1}{2})_1, -1\} - \{(\frac{1}{2}, \frac{1}{2})_{-1}, 1\}$ under $SU(2)\otimes SU(2)$. Equations (1) and (3) then imply a second $SU(2)\otimes SU(2)$ algebra:

$$[Q^i, Q^j] = i\epsilon^{ijk}Q^k, \quad [K^i, Q^j] = i\epsilon^{ijk}K^k,$$
$$[K^i, K^j] = i\epsilon^{ijk}Q^k. \tag{4}$$

Furthermore, the transformation properties [as $(\frac{1}{2}, \frac{1}{2})$] of K^i imply that

$$[K^i, Q_5{}^j] = i\delta^{ij}S, \tag{5}$$

where S is an isoscalar, so that

$$[S, Q^i] = 0. \tag{6}$$

By the Jacobi identity,

$$[Q_5{}^i, S] = iK^i \text{ and } [K^i, S] = -iQ_5{}^i, \tag{7}$$

closing the algebra of Q^i, $Q_5{}^i$, K^i, and S on that of Sp(4) or O(5). The operator V is then $e^{-i\alpha S}$. [In the case of $SU(3)\otimes SU(3)$ the algebra is larger and instead of O(5) we obtain an SU(6) as will be discussed elsewhere.]

Equation (3), even if only an approximation to a more complicated form, may be of great use phenomenologically as it correlates many otherwise unrelated quantities. The power of Eq. (3) in making many predictions results because (a) Q_5, as a *generator* of $SU(2)\otimes SU(2)$, has known matrix elements and in particular can only connect a given irreducible representation with itself; (b) K can only connect different representations with different values of L_z and S_z.

We now proceed to explore these predictions, considering first the results which follow from matrix elements where K cannot contribute, and which, therefore, depend on Q_5 and its property of being a generator. For example, since the nucleon and $N^*(1236)$ with $J_z = \frac{1}{2}$ lie in the representation $\{(1, \frac{1}{2})_{1/2}, 0\}$ of quark constituents, their constituent states can only be connected by the first term in (3b). Using (2), we immediately obtain from taking the one-nucleon matrix element of Q_5 that

$$g_A = \tfrac{5}{3}\cos\alpha, \tag{8}$$

which gives $\cos\alpha = 0.745 \pm 0.005$.[9] This value fixes the relative scale of Q_5 and K for all processes. All matrix elements due to Q_5 will now be reduced by the factor $\cos\alpha$ from their "quark constituent" values.

To proceed further with experimental comparisons, we must use partial conservation of axial-vector currents (PCAC)[10] to relate matrix elements of Q_5 to those of the pion.[11] For example, from $\langle N|Q_5|N^*(1236)\rangle$ we obtain[12]

$$g^* = \tfrac{4}{3}\cos\alpha = 0.98, \tag{9}$$

which is in satisfactory agreement[4] with experiment if we use PCAC to relate g^* to the $N^*(1236) \to N\pi$ amplitude. Extended to the $\frac{1}{2}^+$ octet, we obtain the standard value of $F/D = \frac{2}{3}$.

Since Eq. (3) is an operator statement, we can also take its matrix elements between meson states. For the $q\bar{q}$ states with $L = 0$, the $J_z = 0$, ρ and π are in $(1, 0)_0 \pm (0, 1)_0$ and connected by Q_5. Equation (3) plus PCAC immediately gives

$$g_{\rho\pi\pi} = \sqrt{2}(m_\rho/f_\pi)\cos\alpha, \tag{10}$$

and therefore $\Gamma(\rho \to \pi\pi) \simeq 150$ MeV, in excellent agreement with experiment.[9] Similarly, for $J_z = 1$ the ρ and ω are in $\{(\frac{1}{2}, \frac{1}{2})_1, 0\}$, and Eq. (3) plus PCAC gives

$$g_{\rho\omega\pi} = (\sqrt{8}/f_\pi)\cos\alpha. \tag{11}$$

Within the large uncertainties in extracting $g_{\rho\omega\pi}$ from $\omega \to \pi^0\gamma$ using vector dominance, Eq. (11) is also in very adequate agreement with experiment.[13]

Encouraged by these results for matrix elements of the $Q_5\cos\alpha$ term in Eq. (3), we consider the $L = 1$ meson states of the quark model. We label the $I = 1$ states with $J^{PC} = 1^{+-}$, 2^{++}, 1^{++}, and 0^{++} as B, A_2, A_1, and δ, respectively, while their $I = 0$ counterparts *composed of nonstrange quarks* are labeled H, f, D, and σ. Besides untestable relations involving pionic transitions between $L = 1$ states, we arrive at a number of relations for transitions of $L = 1$ to $L = 0$ mesonic states which proceed *only* through K (Q_5 being a generator does not contribute)[14]:

(1) For $J_z = 0$, $g_{B\omega} = 0$ since both B and ω have $L_z = 0$ and K has $L_z = \pm 1$. Similarly, $g_{H\rho} = 0$ for $J_z = 0$.

(2) For $J_z = 1$, $g_{B\omega}/g_{A_2\rho} = \sqrt{2}$. Using experimental values[9] for $\Gamma(A_2 \to \rho\pi)$ and masses yields $\Gamma(B \to \omega\pi) \simeq 75$ MeV with a purely transverse de-

cay. The width agrees with experiment,[9] where the decay also appears to be dominantly transverse.[15]

(3) In $SU(6)_W$, the pion transforms like Q_5, while no term like K is present. The $J_z=1$ decays of the B and H are then forbidden and the $J_z=0$ decays allowed[16]; our scheme predicts the opposite.

(4) For $J_z=1$, $g_{A_1\rho}/g_{A_2\rho}=1$, and for $J_z=0$, $g_{A_1\rho}/g_{f\pi}=\sqrt{3}$. Assuming $M_{A_1}=1070$ MeV and using experimental data,[9] we obtain $\Gamma(A_1\rightarrow\rho\pi)\simeq 85$ MeV and a dominantly longitudinal decay.[17] This relatively narrow resonance is presumably not to be identified with the wide nonresonance observed[18] in $\pi p\rightarrow(3\pi)p$.

(5) $g_{\delta\eta}/g_{A_2\eta}=\sqrt{2}$. Identifying the δ with the proposed state[9] near 975 MeV, we calculate from $\Gamma(A_2\rightarrow\pi\eta)$ that $\Gamma(\delta\rightarrow\pi\eta)\simeq 35$ MeV. This disagrees with a very narrow state, but agrees with experiments observing the $\pi\eta$ mode of the δ.[19]

(6) $g_{\sigma\pi}/g_{f\pi}=\sqrt{2}$. If we assign, somewhat arbitraily, the nonstrange quark σ meson to have $m_\sigma=m_\rho$, then $\Gamma(f\rightarrow\pi\pi)$ yields $\Gamma(\sigma\rightarrow\pi\pi)\simeq 250$ MeV.[9] The width depends strongly on m_σ. A transition through an operator transforming like Q_5 as in $SU(6)_W$ results in an unacceptable $\Gamma(\sigma\rightarrow\pi\pi)$ of ~60 MeV.

In trying to extend the results to $L=2$ mesons, we find many relations, but almost no presently testable ones. One relation which is of interest is

$$\frac{\langle I=1, J^{PC}=1^{--}|Q_5|\pi\rangle}{\langle I=1, J^{PC}=3^{--}|Q_5|\pi\rangle}=\left(\frac{3}{2}\right)^{1/2}.$$

Identifying the $J^{PC}=3^{--}$ state with the g meson[9] and assuming a mass of 1500 MeV for the $J^{PC}=1^{--}$ state (ρ') yields a sizable width (~150 MeV) into two π's for the ρ'. However, if the ρ' was a radial excitation in the quark model, then its decay into two π's is forbidden by Eq. (3b) and the fact that Q_5 is a generator. Thus if the ρ' state observed[20] at 1500–1600 MeV is a mixture of the $L=2$ state with a dominant $L=0$ radial excitation, its 2π decay is suppressed. In this case the $\rho\pi$ decay mode of its isoscalar companion, ω', is also suppressed.

We may use our results to investigate the contributions to the Adler-Weisberger sum rules for meson targets. For each of the eleven sum rules for $I=1$ meson targets with $L=0$ and 1, $\cos^2\alpha$ (55%) of the total sum rule is fixed in a known way as arising from the Q_5 term in Eq. (3b), while each nonexotic intermediate state contributes positively to the remaining $\sin^2\alpha$ (45%) of the total sum. It is entirely nontrivial that the sum of the remaining identifiable contributions to each of the sum rules does not exceed the 45% limit. For example, the $\pi\pi$ sum rule gives us an upper bound for the contribution of the f meson, yielding $\Gamma(f\rightarrow\pi\pi)\lesssim 180$ MeV.

For the $L=1$ baryons, it is difficult to find simple testable predictions because several of the physical states are presumably mixtures[21] of constituent-quark spin doublet or quartet states, each with its own matrix element for decay to πN or $\pi N^*(1236)$. There still is one relation, $\langle S_{31}|Q_5|N\rangle/\langle D_{33}|Q_5|N\rangle=\sqrt{2}$, between the presumably unmixed spin doublet $I=\frac{3}{2}$ states, S_{31} and D_{33}. Present data on the elastic widths[9,22] are on the borderline of disagreement with this relation, but the errors are fairly large and can accommodate it.[23]

A possibly more serious difficulty with the simple form in Eq. (3) is that any classification of the Roper resonance, $P_{11}(1470)$, as a radial excitation in the quark model, results in its πN and $\pi N^*(1236)$ decay modes being forbidden. These transitions are not large on the scale of $N^*(1236)\rightarrow N\pi$ or $\rho\rightarrow\pi\pi$. Their presence indicates either an unpleasant classification of the P_{11} or the presence of additional terms in the transformed Q_5 which were neglected in Eq. (3).

Note that our results for baryons are not a special case of Ref. 4; contrary to what is assumed there, we must mix the $N^*(1236)$. We also differ in having relations between baryon and meson decays and in our results for the decay of higher states which do not require new parameters or saturation assumptions.

Taken all together, our results are encouraging, there being no great contradictions and several predictions which are in good agreement with experiment. The success of our predictions indicates that Eq. (3) may be an excellent first approximation to the actual case, with a rather elegant form, simple properties, and easily derivable consequences. We hope to report on the details of the above results as well as on considerations of mass formulas and the extension of the scheme to current densities (e.g., magnetic moments) elsewhere.

---

*Work supported by the U. S. Atomic Energy Commission.

†On leave of absence from the Weizmann Institute of Science, Rehovot, Israel.

[1] M. Gell-Mann, Phys. Rev. 125, 1067 (1972), and

Physics (Long Is. City, N. Y.) 1, 63 (1964).

[2]S. L. Adler, Phys. Rev. Lett. 14, 1051 (1965); W. I. Weisberger, Phys. Rev. Lett. 14, 1047 (1965).

[3]For a disucssion, see S. L. Adler and R. F. Dashen, *Current Algebras* (Benjamin, New York, 1968).

[4]For a review, H. Harari, in *Spectroscopic and Group Theoretical Methods in Physics* (North-Holland, Amsterdam, 1968), p. 363, and reference to previous work therein.

[5]Suggested by R. F. Dashen and M. Gell-Mann [Phys. Rev. Lett. 17, 340 (1966)] in connection with the local algebra. F. Buccella *et al.* [Nuovo Cimento 69A, 133 (1970), and 9A, 120 (1972)] suggest a phenomenological scheme for charges.

[6]The extension to $SU(3)\otimes SU(3)$ is straightforward. See Ref. 4.

[7]H. J. Melosh, unpublished. We thank M. Gell-Mann and H. J. Melosh for several informative discussions.

[8]$V^{-1}Q^iV=Q^i$ because isospin is conserved.

[9]P. Söding *et al.*, Phys. Lett. 39B, 1 (1972).

[10]We use $f_\pi=135$ MeV from the π decay amplitude.

[11]Intrinsic to the use of PCAC is an ~10% error. All widths are calculated in narrow-resonance approximation, assuming PCAC for the Feynman amplitude and using phase space for massive π's.

[12]We define $g_{AB}=\langle A|Q_5|B\rangle$, where A and B are physical states. g^* is defined in Ref. 4.

[13]From the model of M. Gell-Mann *et al.* [Phys. Rev. Lett. 8, 261 (1962)] and experimental widths, we obtain $g_{\rho\pi\omega}=(14.4\pm1.0)$/GeV using $\gamma_\rho^2/4\pi=0.6$. Equation (11) gives a value of 15.6/GeV. In addition to the purely experimental errors, there is an unknown error inherent in the model.

[14]Our results for $L=1$ to $L=0$ transitions agree with those of Buccella *et al.*, Ref. 5, but we disagree in general.

[15]See the recent work of R. Ott, thesis, University of California, Berkeley, 1972 (unpublished), and earlier references therein.

[16]See the references and discussion of the $SU(6)_W$ predictions and their breaking by E. W. Colglazier and J. L. Rosner, Nucl. Phys. B27, 349 (1971).

[17]F. J. Gilman and H. Harari, Phys. Rev. Lett. 18, 1150 (1967), and Phys. Rev. 165, 1803 (1968).

[18]See, for example, R. Klanner, in *Experimental Meson Spectroscopy—1972*, AIP Conference Proceedings No. 8, edited by A. H. Rosenfeld and K. W. Lai (American Institute of Physics, New York, 1972), p. 164.

[19]This modifies slightly the analysis contained in Ref. 17, where $\delta\not\to\eta\pi$.

[20]H. H. Bingham *et al.*, Phys. Lett. 41B, 635 (1972), and references to other experiments therein.

[21]Large mixing is needed in $SU(6)_W$. See D. Faiman and D. E. Plane, CERN Report No. CERN-Th-1549, 1972 (unpublished).

[22]See also the recent analysis of R. Ayed *et al.*, unpublished.

[23]A particular choice of parameters in broken $SU(6)_W$ gives results which agree with ours. See Ref. 16 and W. P. Petersen and J. L. Rosner, Phys. Rev. D 6, 820 (1972). We thank J. L. Rosner for discussions and pointing out an error in an earlier manuscript.

ERRATUM

MASS FORMULA FOR KERR BLACK HOLES. Larry Smarr [Phys. Rev. Lett. 30, 71 (1973)].

Dr. Robert V. Penney has pointed out an algebraic error in the transformation of parameters from A, L, Q to η,β,ϵ (page 72) in the quantities E_r and $E_{\rm em}$. These two lines should be changed to

$$E_r=\tfrac{1}{2}\eta[(1-\beta_0^2)^{-1/2}-1],$$

$$E_{\rm em}=\tfrac{1}{2}\eta[(1+\epsilon^2)(1-\beta^2)^{-1/2}-(1-\beta_0^2)^{-1/2}],$$

where

$$\beta_0=\beta(A,\,L,\,Q=0).$$

Further, the line giving the second-order expansion of $E_{\rm em}$ should read

$$E_{\rm em}\cong\tfrac{1}{2}Q^2\eta^{-1}.$$

The conclusion of the paper remains unaltered.