Note on the Sato-Tomimatsu Solution of Einstein's Equations

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An examination of the Sato-Tomimatsu solution shows it to represent a spinning naked singularity with causality violation and a singular ergosphere. It does not contain an event horizon.

Recently Sato and Tomimatsu[1] have published a new, stationary axisymmetric, asymptotically flat exact solution to Einstein's vacuum field equations. Until now the only known solutions of this sort have been the Kerr solution,[2] which represents a black hole, and the Papapetrou solutions,[3] which are massless. The authors have claimed that their solution contains an event horizon. This paper contains a brief account of our investigation of this solution in which we show that no event horizon is present—the solution represents a spinning naked singularity. Because of its algebraic complexity we have used a computer to perform much of the algebra, numerically evaluate roots, and compute the curvature tensor, using the Cambridge algebra system (CAMAL).[4]

The solution is of the form

$$ds^2 = Bp^{-4}(x^2-y^2)^{-4}(dz^2+d\rho^2)+\rho^2BA^{-1}d\varphi^2 - AB^{-1}[dt-2mqA^{-1}C(1-y^2)d\varphi]^2,$$

with

$$A=[p^2(x^2-1)^2+q^2(1-y^2)^2]^2-4p^2q^2(x^2-1)(1-y^2)(x^2-y^2)^2,$$

$$B=(p^2x^4+q^2y^4-1+2px^3-2px)^2+4q^2y^2(px^3-pxy^2+1-y^2)^2,$$

$$C=p^2(x^2-1)[(x^2-1)(1-y^2)-4x^2(x^2-y^2)]-p^3x(x^2-1)[2(x^4-1)+(x^2+3)(1-y^2)]+q^2(1+px)(1-y^2)^3,$$

and

$$\rho^2=m^2p^24^{-1}(x^2-1)(1-y^2),\quad z^2=m^2p^24^{-1}x^2y^2,\quad p^2+q^2=1.$$

The mass is m and the angular momentum m^2q^2. If $q=0$ the solution reduces to a Weyl solution discussed in Voorhees.[5] We shall restrict ourselves to considering $x\geqslant 1$.

Ring singularity.—As pointed out in Ref. 1 the solution has a ring singularity when the norm of the timelike Killing vector becomes infinite on the equator at the root of $B(x, y=0)=0$.

Ergosphere.—The timelike Killing vector becomes null at the roots of A. These are given by

$$x^2=1+\lambda^2(1-y^2),$$

where λ is the root of

$$p^2\lambda^4+q^2-4pq\lambda(\lambda^2+1)=0.$$

There are two roots for $x\geqslant 1$, the smaller of which coincides (on the equator) with the ring singularity. However, since A has a single root and B a double root this does not cancel out the singularity.

Causality violation.—If we interpret φ as the angular coordinate, as we must to preserve the asymptotic flatness, then if $g_{\varphi\varphi}$ is negative we have closed timelike loops. We find after some manipulation that

$$g_{\varphi\varphi}=m^2(1-y^2)4^{-1}B^{-1}F,$$

where F is a polynomial of degree 10 in x and 6 in y. We find that it has one root for $x>1$ which for $y=0$ coincides with the ring and for $y^2=1$ coincides with the "poles" of the surface $x=1$. For intermediate values it lies inside the inner ergosphere.

The nature of the surface $x=1$.—Sato and Tomimatsu claim, by analogy with the Kerr solution, that $x=1$ is an event horizon. This is not so since the induced (degenerate) metric is Lorentzion there. This could not happen of $x=1$ were a null surface. It is easy to check that the two

Killing vectors $\partial/\partial\varphi$ and $\partial/\partial t$ become parallel at $x=1$. We have computed the curvature tensor in the "obvious" orthonormal frame at $x=1$, and it is finite as it is in the coordinate frame. Thus presumably the solution is extendible through $x=1$. One possible extension is to make $x=1$, $y^2<1$ part of an axis. This involves reidentifying the φ coordinate which spoils the axis for $x>1$, $y^2=1$, and makes t a periodic coordinate which spoils the asymptotic flatness. Similar behavior occurs in Hartle and Hawking[6] or in Taub-Newman-Unti-Tamburino space. An examination of the equatorial geodesics indicates that they will meet $x=1$ in finite affine parameter time, so the solution is clearly geodesically incomplete and thus a naked singularity.

The axis.—The induced metric on the axis $x \geq 1$, $y^2=1$ is of the general form given in Walker.[7] An examination of the geodesics lying in the axis shows that the spacelike geodesics meeting $x=1$ are infinitely long; the timelike and null ones, however, are of finite length and the methods of Ref. 7 can be used to extend the axis considered as a two-manifold. This "directional behavior" is similar to that exhibited in the $q=0$ case.[5]

Petrov type.—Using the methods of D'Inverno and Russell-Clark[8] we find that the solution is of the general Petrov type, i.e., there are four distinct principal null directions.

Discussion.—The properties we have described above would seem to indicate that the astrophysical application of these solutions is small since the regions inside the outer ergosphere are pathological. It is possible that the solution will find application as the exterior field of a very special rotating star, with a special relationship between all of its multipole moments. It has no relevance to black-hole physics.

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Pion-Pion Phase Shift and Form Factors in the Decays $K^{\pm} \to \pi^+\pi^- e^{\pm}\nu_e$

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We present the results of an analysis of 8141 K_{e4} decays. These include (i) the form factors of the weak hadronic current and their dependences on the invariant di-lepton and di-pion masses, and (ii) the magnitude and dependence on invariant di-pion mass of the π-π scattering phase shift $\delta_s - \delta_p$.

It has been recognized for about a decade that the decays $K \to \pi\pi e\nu$ (K_{e4}) provide the possibility of a direct and unambiguous determination of pion-pion phase shifts over a range of energies inaccessible to other methods of measurement.[1] This possibility, which arises because of the absence of other hadrons in the final four-body state, has stimulated much effort to describe the structure of the decay distribution[2-4] in such a way as to permit the π-π phase shifts to be simply evaluated with a minimum of assumptions.[5] It has also stimulated several attempts to measure the decay distribution.[6-8]

In an earlier note[9] we presented a preliminary value of the pion-pion phase shift $\delta_s - \delta_p$, where δ_s and δ_p are the π-π scattering phase shifts in the di-pion states with $I=0$, $l=0$, and $I=1$, $l=1$, respectively. We describe here a more detailed analysis of 8141 K_{e4} events from which we extract (i) the form factors of the weak hadronic current and their dependences on the invariant di-lepton and di-pion masses, $M_{e\nu}$ and $M_{\pi\pi}$; and