Regge-Cut Discontinuities and Elastic Unitarity*

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We show that elastic unitarity does not require the discontinuities of Regge cuts to vanish at the branch point when the momentum transfer is below the crossed-channel elastic threshold. This is in contrast to the conclusions of Bronzan and Jones.

Part of the lore on the theory of Regge cuts is the belief that the discontinuity of a scattering amplitude across a cut in the complex angular momentum (l) plane should vanish as the tip of the cut is approached. Using elastic unitarity in the crossed channel, Bronzan and Jones[1] concluded that this behavior, which we refer to as "soft," is a general property of all Regge branch points.

In this paper we re-examine the constraints imposed on Regge cuts by crossed-channel (s-channel) elastic unitarity. We find that elastic unitarity does not eliminate hard cuts, i.e., ones with nonvanishing discontinuities at the tip, if the crossed-channel total four-momentum squared (s) is less than the elastic threshold (s_e). Thus, in the region of relevance to the asymptotic scattering in the physical direct channel, hard cuts are allowed. The fact that the Bronzan-Jones condition can fail at isolated values of s has been noted previously.[2] We show that hard cuts can occur for a continuous range of momentum transfer.

Whenever s lies between s_e and the inelastic threshold (s_i), unitarity does constrain the form of Regge cuts ending at real $l=\alpha_c(s)$. In this case we find that if the magnitude of the discontinuity does not vanish at the cut tip, then the continued partial-wave phase shift must diverge at the branch point. With such a behavior hard cuts may occur here as well.[3]

We will demonstrate our assertions by first giving an example of a unitary, continued partial-wave amplitude having at $l=\alpha_c(s)$ a cut which is hard for $s<s_e$, but which becomes soft for $s>s_e$. Then we will discuss the connection between hard cuts for $s_i>s>s_e$ and divergent phase shifts.

We use the notation of Bronzan and Jones[1] in what follows. Elastic s-channel unitarity continued to complex angular momentum is automatically satisfied if we write the amplitude in the form

$$B(s,l)=[W(s,l)+Y(s,l)]^{-1}, \tag{1}$$

where

$$W(s,l)=[4^{l+1/2}s^{1/2}\cos(\pi l)]^{-1}(s_e-s-i\epsilon)^{l+1/2}. \tag{2}$$

Here $Y(s,l)$ is a real analytic function of s for real l except for left-hand cuts, inelastic cuts, and Regge branch points. We consider a single Regge cut at $l=\alpha_c(s)$ and let $s_c(l)$ be the s-plane position of this cut. We make the standard assumption that $\alpha_c(s)$ is a real analytic function of s with cuts for $s\geqslant s_i$. We also assume that $\alpha_c(s)$ monotonically increases with real $s<s_i$. For real l such that $s_c(l)<s_i$ we can write the discontunuity of $B(s,l)$ across the s-plane Regge cut beginning at $s_c(l)$ as[1]

$$\gamma(s,l)=\frac{-2i\,\mathrm{Im}Y(s+i\epsilon,l)}{[W(s+i\epsilon,l)+\mathrm{Re}Y(s+i\epsilon,l)]^2+[\mathrm{Im}Y(s+i\epsilon,l)]^2}. \tag{3}$$

As was pointed out by Bronzan and Jones,[1] γ must vanish if $\mathrm{Im}Y$ approaches a constant or becomes infinite at the branch point. However, γ need not vanish at the end of the cut if $\mathrm{Im}Y(s+i\epsilon,l)$ goes to zero there and $W(s+i\epsilon,l)+\mathrm{Re}Y(s+i\epsilon,l)$ also approaches zero as fast or faster. We now give an example of a Y that provides this cancelation between W and $\mathrm{Re}Y$ for a continuous range of values of l such that $s_c(l)<s_e$:

$$Y(s,l)=-W(s_c(l),l)+\frac{1}{\pi}\int_{\alpha_c(s_e)}^{\alpha_c(s_i)}\frac{dl'}{l'-l}\,\mathrm{Im}[W(s_c(l'),l')][s_c(l)-s]^{(l'-l)/a}+C(s,l). \tag{4}$$

Here a is a positive constant and $C(s,l)$ vanishes at $s=s_c(l)$ for $s_c(l)<s_e$ and is analytic in s and l except for left-hand cuts, inelastic cuts, and a possible branch point at $s_c(l)=s$. The integral in Eq. (4)

serves to remove the fixed l-plane cut in $W(s_c(l), l)$ starting at $l = \alpha_c(s_e)$. We cut off the integral at $\alpha_c(s_i)$ because here we expect other singularities to become important. These singularities are beyond the scope of this paper.

To illustrate how this type of cut behaves, assume the following specific form for C:

$$C(s, l) = [s_c(l) - s]^{1/2}. \tag{5}$$

Taking $s_e > s > s_c(l)$, we have

$$\mathrm{Im}Y(s+i\epsilon, l) = \frac{-1}{\pi}\int_{\alpha_c(s_e)}^{\alpha_c(s_i)} \frac{dl'}{l'-l}[\mathrm{Im}W(s_c(l'), l')]\,[s - s_c(l)]^{(l'-l)/a} \sin\left[\pi\frac{l'-l}{a}\right] - [s - s_c(l)]^{1/2}, \tag{6}$$

$$W(s, l) + \mathrm{Re}Y(s+i\epsilon, l) = \frac{1}{\pi}\int_{\alpha_c(s_e)}^{\alpha_c(s_i)} \frac{dl'}{l'-l}[\mathrm{Im}W(s_c(l'), l)](s - s_c)^{(l'-l)/a} \cos\left[\pi\frac{l'-l}{a}\right] + O(s_c(l) - s). \tag{7}$$

When l is less than $\alpha_c(s_e) - \frac{1}{2}a$, the square-root term in $\mathrm{Im}Y$ dominates as $s \to s_c(l)$; consequently, $B(s, l)$ has essentially an inverse square-root cut. This is an example of a hard cut. As l becomes greater than $\alpha_c(s_e) - \frac{1}{2}a$ the integral terms in Eqs. (6) and (7) become important, and the cut in $B(s, l)$ begins to soften. When l exceeds $\alpha_c(s_e)$, the integral in Eq. (4) defining Y becomes infinite as $s \to s_c(l)$. Because of Eq. (1), the cut is now truly soft with a discontinuity vanishing at the endpoint.

By choosing different $C(s, l)$ we can obtain a wide variety of different behaviors for γ. As another example, taking

$$C(s, l) = 1/\ln[s_c(l) - s] \tag{8}$$

gives a discontinuity in B that approaches a constant at the branch point for $s_c(l) < s_e$. Note, however, that on unphysical Riemann sheets associated with the elastic branch point these cuts will not be hard because the delicate cancelation required between $\mathrm{Re}Y(s, l)$ and $W(s, l)$ will not occur. In general it is not possible for the cut to be hard on all such sheets.[1]

The above examples give soft cuts when $s_c(l) > s_e$. However it is possible to have hard cuts for such $s_c(l)$ as well if we allow the phase shift for the amplitude to diverge as the cut is approached. To see this we study the continued partial-wave phase shift defined by

$$B(s, l) = \frac{\sqrt{s}\,4^{l+1/2}}{(s - s_e)^{l+1/2}} e^{i\delta(s,l)} \sin\delta(s, l). \tag{9}$$

Unitarity tells us that $\delta(s, l)$ is real for real l and $s_e < s < s_c(l) < s_i$. Unless $B(s, l)$ has an accumulation of other singularities at $s_c(l)$, $\delta(s, l)$ is continuable as a real analytic function of s in some neighborhood (N) of $s_c(l)$ with a cut running to the right from $s_c(l)$. In this neighborhood we can write the Cauchy representation

$$\delta(s, l) = \frac{1}{\pi}\int_{s_c(l)}^{s_N} \frac{ds'}{s' - s - i\epsilon}\,\mathrm{Im}(\delta(s+i\epsilon, l)) + \frac{1}{2\pi i}\int_C \frac{ds'}{s'-s}\,\delta(s, l), \tag{10}$$

where $s_N > s_c(l)$ is in N and C is any simple closed contour in N passing through s_N and enclosing both s and the portion of the cut between $s_c(l)$ and s_N. Now we can calculate the Regge cut discontinuity:

$$\gamma = \mathrm{disc}B(s, l) = R(s, l)\exp\left[\frac{2i}{\pi}\,\mathrm{P}\int_{s_c(l)}^{s_N}\frac{ds'}{s'-s}\,\mathrm{Im}\,\delta(s+i\epsilon, l)\right]\sinh[2\,\mathrm{Im}\,\delta(s, l)], \tag{11}$$

where $R(s, l)$ is a nonvanishing regular function for s near $s_c(l)$. Clearly $|\gamma|$ approaches a nonzero limit as $s \to s_c(l)$ if and only if $\mathrm{Im}\,\delta(s, l)$ also has a nonvanishing limit. But if $\mathrm{Im}\,\delta(s, l)$ does not vanish at $s_c(l)$, then the integral giving the phase of γ will diverge as $s \to s_c(l)$. This shows the required divergence of the phase shift. Writing the amplitude in the form of Eq. (1), we find that this divergence in the phase shift corresponds to an accumulation of poles (Castillejo-Dalitz-Dyson poles) about $s_c(l)$ in the function $Y(s, l)$. Such a possibility was not considered in the paper of Bronzan and Jones.[1,3]

In summary, we have shown that the common assumption that a Regge cut discontinuity must vanish at the cut tip does not follow from elastic unitarity. This means that models for Regge cuts that violate this condition cannot be immediately eliminated.[4] Regge cuts are constrained by crossed-channel elastic unitarity only when the momentum transfer is in the physical region for the crossed-channel scattering. These constraints are of no relevance to the asymptotic behavior of the direct-channel physical scattering.

Note added in proof.—After this work had been completed, we received a paper by Branson[5] pointing out some specific mechanisms which allow hard Regge cuts.

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[1]J. B. Bronzan and C. E. Jones, Phys. Rev. 160, 1494 (1967).

[2]I. J. Muzinich, F. E. Paige, T. L. Trueman, and L.-L. Wang, Phys. Rev. Lett. 28, 850 (1972).

[3]J. B. Bronzan (private communication) has independently discovered this possibility. Also, see J. B. Bronzan, to be published.

[4]Some examples of such models are given by S. Auerbach, R. Aviv, R. L. Sugar, and R. Blankenbecler, Phys. Rev. D 6, 2216 (1972); H. Cheng and T. T. Wu, Phys. Rev. Lett. 24, 1456 (1970); J. Finkelstein and F. Zachariasen, Phys. Lett. 34B, 631 (1971).

[5]D. Branson, to be published.

Two-Photon Exchange in Electron-Deuteron Scattering*

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We show that multiple scattering of the electron (i.e., two-photon exchange) in electron-deuteron elastic scattering leads to a small component in the cross section, which decreases very slowly with momentum transfer and may dominate the cross section at high momentum transfer.

Measurements of electron-deuteron elastic scattering have, over the years, been pushed to higher and higher values of momentum transfer. Such experiments are a means of exploring the small-distance electromagnetic structure of the deuteron, and in the most recent series reached momentum transfers squared of 1.4 GeV^2, or distances of about $\frac{1}{6}$ fm.[1] An essential ingredient in the analysis of such measurements is the assumption that the process is of the lowest relevant order in the electromagnetic interaction, that is, that the matrix element is proportional to a photon propagator times form factors which depend only on the structure of the deuteron. In the case of electron-proton scattering this assumption has been tested by searching for effects that would arise from a two-photon–exchange contribution.[2]

In this note we would like to point out that electron-deuteron elastic scattering offers conditions where two-photon exchange may become significant or perhaps dominant. This possibility is due to the existence at high-momentum transfer of the simple mechanism of successive scatterings on the two nucleons. This mechanism has been observed in pion-deuteron and proton-deuteron elastic scattering and even in ρ photoproduction,[3] and is calculable in terms of known quantities in a relatively simple manner.

In general the amplitude for an elastic-scattering process on the deuteron can be decomposed into two terms, as in the method developed by Glauber.[4] The first, the single-scattering term, is large but drops rapidly with momentum transfer, since it represents coherence over the entire spatial extent of the deuteron. The second, the double-scattering term, drops much more slowly with momentum transfer since the neutron and the proton can each take half the transferred momentum and move off together, recombining to form the deuteron. While the room available for this effect in the n-p phase space will clearly be small, its decrease with t is rather slow, depending only on the structure of the constituents. Thus it can eventually overtake the single-scattering contribution. In electron-deuteron scattering naturally, the double-scattering cross section is intrinsically smaller by a factor of $\alpha^2=(1/137)^2$. We show below that it may, nevertheless, be comparable to the one-photon exchange at values of momentum transfer t of the order of a few GeV^2.