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[11]It is clear from the discussion in Ref. 5, which concedes the complicated band structure of Ni and the apparent inconsistency between the Gd thermopower results (as interpreted by the static entropy model) and neutron diffraction measurements of the itinerant magnetic moment in Gd, that the empirical support for the static entropy model is not nearly as convincing as it appears to be in Refs. 3 and 4.

[12]While the spin scattering model can account for similarities in the functional form of C_e and $T\,dQ/dT$, it cannot account for prefactor similarities (weighted by the degree of magnetic itineracy), which are explicitly required in the static entropy model.

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Inequality Relating the Ground-State Energies of Two and Three Bosons

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We report sufficient conditions for the validity of the inequality $E_0(3) \le 3E_0(2)$ for the ground-state energies of two and three nonrelativistic identical bosons interacting via spherically symmetric pair potentials.

Let $E_0(2)$ and $E_0(3)$ be the ground-state energies for two and three nonrelativistic identical bosons interacting by spherically symmetric pair potentials. We have established sufficient conditions for the relation

$$E_0(3) \leq 3E_0(2) \tag{1}$$

to be valid. Our conditions are stated in terms either of properties of the pair potential $V(r)$ or of properties of the ground-state solution[1] $\varphi(r)$ of the two-body Schrödinger equation for particles of mass m,

$$-(\hbar^2/m)\nabla^2\varphi + V(r)\varphi = E_0(2)\varphi. \tag{2}$$

Any of the following conditions are sufficient to prove Eq. (1). (i) $V(r)$ has one minimum and no other local extrema; that is, there is some separation R (R may be zero) for which the following conditions are valid:

$$\begin{aligned} &dV/dr \leq 0, \quad 0 \leq r \leq R; \\ &dV/dr \geq 0, \quad R \geq r < \infty. \end{aligned} \tag{3}$$

(ii) $F(r)$ defined by

$$F(r) \equiv r\varphi^2(r) \tag{4}$$

has one maximum. (iii) $F(r)$ defined by Eq. (4) satisfies the following conditions:

$$\begin{aligned} &F(r) = 0, \quad r \leq \sigma; \\ &dF/dr \leq 0, \quad r \geq 2\sigma. \end{aligned} \tag{5}$$

The proofs are based on the use of the Jastrow-type trial function for three particles,

$$\psi(1,2,3) = \varphi(r_{12})\varphi(r_{23})\varphi(r_{31}) \tag{6}$$

(the r_{ij} are the interparticle separations), in a Rayleigh-Ritz upper bound for $E_0(3)$. We first obtain

$$E_0(3) \leq 3E_0(2) - (3\hbar^2/4m)\int d^3r_1\, d^3r_2\, d^3r_3\, \varphi^2(r_{23})[\nabla_1\varphi^2(r_{12})]\cdot[\nabla_1\varphi^2(r_{13})] \times \left[\int d^3r_1\, d^3r_2\, d^3r_3\, \varphi^2(r_{23})\varphi^2(r_{12})\varphi^2(r_{13})\right]^{-1}. \tag{7}$$

After some algebra we obtain

$$E_0(3) - 3E_0(2) \leq (9\hbar^2/8m)\,J/N, \tag{8}$$

where

$$J = \int_0^\infty dr \int_0^\infty ds\, F(r)F(s)(d/dr)F(r+s), \quad N = \int_0^\infty dr \int_0^\infty ds \int_{r-s}^{r+s} dt\, F(r)F(s)F(t). \tag{9}$$

The manipulations to obtain this form for J include the following chain of equalities:

$$J=\tfrac{2}{3}\int_0^\infty dr\,F(r)\int_0^r ds\,F(s)\int_{r-s}^{r+s}dt\,d^2F(t)/dt^2$$

$$=\tfrac{1}{3}\int_0^\infty dr\int_0^\infty ds\,F(r)F(s)(d/dr)F(r+s)-\tfrac{2}{3}\int_0^\infty dr\int_r^\infty ds\,F(r)F(s)(d/ds)F(s-r)$$

$$=\int_0^\infty dr\int_0^\infty ds\,F(r)F(s)(d/dr)F(r+s).$$

We prove Eq. (1) by showing that J is nonpositive for conditions (i), (ii), or (iii).

Under the assumptions of Eqs. (5), the only nonvanishing contribution to the integrals in J occurs for $r+s$ larger than 2σ and therefore J is nonpositive.

Under the assumptions of condition (ii), F is nonnegative and there is a separation L such that the following inequalities are valid:

$$dF/dr\geqslant 0,\quad r\leqslant L,\qquad dF/dr\leqslant 0,\quad r\geqslant L. \tag{10}$$

Then we obtain the following chain of inequalities:

$$J\leqslant\int_0^L dr\,F(r)\Big[\int_0^{L-r}+\int_{L-r}^{L}+\int_L^\infty\Big]ds\,F(s)\,dF(r+s)/ds$$
$$\leqslant\int_0^L dr\,F(r)\{F(L-r)[F(L)-F(L-r)]+F(L-r)[F(L+r)-F(L)]-\tfrac{1}{2}[F(L+r)]^2\}\leqslant 0, \tag{11}$$

and therefore we show that J is nonpositive. To apply condition (i) we define

$$\varphi(r)=u(r)/r,\quad u\geqslant 0, \tag{12}$$

where the function $u(r)$ satisfies

$$d^2u/dr^2=(m/\hbar^2)[V(r)-E_0(2)]u(r). \tag{13}$$

Then using the identity

$$\frac{1}{r(r+s)}+\frac{1}{s(r+s)}=\frac{1}{rs} \tag{14}$$

and the symmetry of the integrals in J, we obtain

$$J=\int_0^\infty dr[u^2(r)/r]\int_0^\infty ds\,2[u(s)u(r+s)/(r+s)^2][u(s)\,du(r+s)/d(r+s)-u(r+s)\,du(s)/ds]$$
$$=\int_0^\infty dr[u^2(r)/r]\int_0^\infty ds\,2[u(s)u(r+s)/(r+s)^2]K(r,s), \tag{15}$$

where $K(r,s)$ is defined by

$$K(r,s)=\int_s^\infty dt\,(m/\hbar^2)[V(t)-V(r+t)]u(t)u(r+t). \tag{16}$$

We have immediately

$$K(r,0)\leqslant 0;\quad K(r,s)\leqslant 0,\quad s\geqslant R \tag{17}$$

using Eqs. (3) and the Schrödinger equation. Further, the minimum of $K(r,s)$ for given r is located at S_0 which is the root of

$$V(S_0+r)=V(S_0). \tag{18}$$

Under the assumptions in Eq. (3) there may be a continuous range of S_0 satisfying Eq. (18); for S in this range K is constant, and for S outside this range K increases. Even for this degenerate case, K has only a local minimum and never becomes positive, since it would then have a local maximum also. Therefore condition (i) is sufficient to show J is nonpositive also.

These results are valid for potentials with hard cores.

It is at least amusing that Eq. (1) provides a more restrictive upper bound on the ground-state energy $E_0(3)$ of neon and argon inert-gas trimers (Lennard-Jones 12-6 pair potentials) than resulted from a recent variational calculation.[2]

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[1]We assume the integrals $\int_0^\infty dr\, r^2\varphi^2(r)$ and $\int_0^\infty dr\, r\varphi^2$ are finite. The case of a zero-range force must be treated as a limiting case.

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Role of Angular Momentum in Complete Fusion Reactions: $B^{11} + Tb^{159}$, $C^{12} + Gd^{158}$, $O^{16} + Sm^{154}$†

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Measurements are reported for the complete-fusion cross sections in the reactions of $B^{11}+Tb^{159}$, $C^{12}+Gd^{158}$, and $O^{16}+Sm^{154}$ at bombarding energies such that the excitation energy of the Yb^{170} compound nuclei that are formed is 107 MeV. The results indicate that the complete-fusion cross sections are determined by dynamical processes in the entrance channel rather than by the equilibrium properties of the compound system.

There has recently been much interest in the experimental and theoretical determination of complete-fusion cross sections in heavy-ion–induced reactions, motivated primarily by the desire to investigate the feasibility of reactions designed to produce superheavy elements.[1-3]

In a recent paper Blann and Plasil[3] calculate complete-fusion cross sections, as they are operationally defined by current experimental techniques, through the use of an angular-momentum-dependent formalism which accounts for the noncompound portion of total reaction cross sections by considering it to be a type of fission. We report here complete-fusion cross sections which indicate that this point of view cannot be a complete picture of the phenomena that are involved.

Measurements have been made of the complete-fusion cross sections for the following entrance channels leading to the formation of the Yb^{170} compound nucleus with 107-MeV excitation energy:

$$\left.\begin{array}{l} B^{11}(115\text{ MeV})+Tb^{159} \\ C^{12}(126\text{ MeV})+Gd^{158} \\ O^{16}(137\text{ MeV})+Sm^{154} \end{array}\right\} \rightarrow Yb^{170},$$

where the laboratory energies of the incident heavy ions are given in parentheses for each entrance channel.

The data were collected at the Yale University heavy-ion accelerator. Isotopically enriched targets for the C^{12} and O^{16} entrance channels were obtained from Oak Ridge National Laboratory and are 1.0 mg/cm^2 and 0.87 mg/cm^2 thick, respectively. The target of Tb^{159}, an isotope which occurs with 100% natural abundance, was made by evaporating the metal onto a 40-μg/cm^2 carbon foil. The thickness was determined by elastic scattering to be 0.16 mg/cm^2.

Following the technique of Kowalski, Jodogne, and Miller[1] and Natowitz,[2] the complete-fusion cross sections were measured with mica detectors. The pieces of mica were etched prior to bombardment for 3 h in 48% hydrofluoric acid in order to enlarge any pre-existing tracks. Following bombardments, the tracks in the mica detectors were developed by again exposing the mica to 48% hydrofluoric acid for about 30 min. The detectors were then scanned under a microscope and a record was kept of the number of complete-fusion events corresponding to a known