position of the shifted luminescence line (30 K, 28 K)[3,4] relative to the peak of the free exciton line. Some of this discrepancy might be attributable to the width of the exciton line, since it seems that the low-energy threshold would be more appropriate than the peak. A value of Φ (17 K) similar to ours was obtained by Pokrovskii[3] from studies of luminescence intensity with power and temperature under steady-state conditions. Theoretical estimates[9,10] give 20 and 29K. Since the discrepancies we are discussing represent only 20% of the correlation energy, the theoretical numbers cannot be relied upon to such accuracy.

In conclusion, observations of the thresholds and decay patterns provide us with information about the energy parameters of the system and insight into the interaction between the droplets and the exciton gas. The good fit obtained to the unusual decay curves confirms the validity of the droplet model.

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Steady-State Motion of Magnetic Domains

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This paper introduces two integrals which simplify calculation of the dynamic properties of magnetic domains. These integrals, over quadratic functions of the spatial derivatives of the magnetization, yield forces acting on a domain which correspond to the gyroscopic and dissipative terms in the Gilbert equation. The force integral corresponging to the gyroscopic term is found to be even less sensitive to the details of the spin distribution than the dissipative drag integral. Hard magnetic bubble domains are considered as an illustrative example.

Recent papers have described the static spin configuration and anomalous dynamic properties of hard magnetic bubbles.[1-6] This paper presents several relations, derived from the Gilbert equation, which greatly facilitate the calculation of some of the dynamic properties of these and other magnetic domains. The relations are applied to the hard-bubble problem as an example. Cartesian tensor notation is used, with repeated indices being assumed summed and with the totally antisymmetric unit tensor being denoted by e_{ijk}. Field position is denoted by x_i and domain position is denoted by X_i. The magnetization is specified either by its three components M_i, or by the saturation magnetization M_s and the polar angles θ and φ. [The z axis (index = 3) corresponds to $\theta = 0$; the x axis (index = 1) corresponds to $\theta = \pi/2$, $\varphi = 0$, the coordinate system being right handed.]

The Gilbert equation, written in tensor notation and arranged in the form which reads, "The time rate of change of angular momentum minus the torque due to linear dissipative effects minus the torque due to reversible effects is equal to zero," is

$$-\frac{1}{|\gamma|}\frac{dM_i}{dt}+\frac{\alpha}{|\gamma|M_s}e_{ijk}M_j\frac{dM_k}{dt}-e_{ijk}M_jH_k{}^r=0, \tag{1a}$$

where the effective field due to reversible effects is

$$H_i^r = -\delta\rho_E/\delta M_i, \tag{1b}$$

ρ_E denoting the energy density and δ denoting functional variation. Multiplication of (1a) by M_i and contracting on i yields the well-known conclusion that the Gilbert equation describes only systems in which $|M_i|$ is conserved. Additionally, only materials in which M_s is spatially constant will be considered here, so that

$$M_i\, dM_i/dt = 0, \tag{2a}$$

$$M_i\, \partial M_i/\partial x_j = 0, \tag{2b}$$

$$M_i M_i = M_s^2. \tag{2c}$$

Consider now the equivalent field equation,

$$H_i^t = H_i^m + H_i^g + H_i^\alpha + H_i^r = 0, \tag{3a}$$

where H_i^m, H_i^g, and H_i^α are the mutually orthogonal vectors [as a consequence of (2a)]

$$H_i^m \equiv \beta M_i$$

(magnetization equivalent field), (3b)

$$H_i^g \equiv \frac{-1}{M_s^2|\gamma|} e_{ijk} M_j \frac{dM_k}{dt}$$

(gyroscopic equivalent field), (3c)

$$H_i^\alpha \equiv -\frac{\alpha}{M_s|\gamma|}\frac{dM_i}{dt}$$

(dissipative equivalent field). (3d)

Multiplication of (3) by $-e_{ijk}M_k$, summing on i, and renaming indices reproduces (1). Thus, when $|M_i|$ is constrained to M_s, (1) and (3) are equivalent.

It will now be shown that the products (for $a = m, g, \alpha$)

$$f_i^a \equiv -H_j^a\, \partial M_j/\partial x_i \tag{4}$$

are force densities. Note first that as a consequence of (2), $f_i^m = 0$ even if $\beta \neq 0$, so that $a = m$ may be ignored. Since $H_i^t = 0$ at all points, then

$$f_i^t = 0. \tag{5}$$

Dividing the reversible term into internal and external terms yields

$$H_i^r = H_i^{r\,\mathrm{in}} + H_i^{r\,\mathrm{ex}}, \tag{6}$$

where $H_i^{r\,\mathrm{ex}}$ is the applied field. The corresponding force density $f^{r\,\mathrm{ex}}$ is (since $\rho_E^{\mathrm{ex}} = -H_j^{r\,\mathrm{ex}} M_j$)

$$f_i^{r\,\mathrm{ex}} = -H_j^{r\,\mathrm{ex}}, \tag{7a}$$

$$\partial M_j/\partial x_i = (\partial H_j^{r\,\mathrm{ex}}/\partial x_i) M_j. \tag{7b}$$

[Equation (7a) is equivalent to (7b) by integration by parts, the bounding surface being located either outside the magnetic material or so that $M_j H_j^{r\,\mathrm{ex}}$ has the same value on opposing surfaces.] Since the external term is clearly a force density and since it is one term of a zero sum, the other terms may be also identified as force densities.

For steady motion with velocity v_j,

$$M_i = M_i(x_j - X_j), \tag{8a}$$

$$X_j = v_j t, \tag{8b}$$

$$dM_i/dt = -v_j\, \partial M_i/\partial x_j. \tag{8c}$$

By using (3c), (4), and (8c), the gyroscopic force density may be written as

$$f_i^g = \hat{g}_{ij} v_j, \tag{9a}$$

the antisymmetric gyrocoupling tensor $\hat{g}_{ij}$ being defined by

$$\hat{g}_{ij} = \frac{1}{M_s^2|\gamma|} e_{kmn} M_k \frac{\partial M_m}{\partial x_i}\frac{\partial M_n}{\partial x_j} \tag{9b}$$

or

$$f_i^g = e_{ijk} g_j v_k, \tag{9c}$$

the gyrocoupling vector g_i being defined by

$$g_i = -\tfrac{1}{2} e_{ijk}\hat{g}_{jk}$$
$$= \frac{-1}{2M_s^2|\gamma|}\delta_{mnp}{}^{ijk} M_m \frac{\partial M_n}{\partial x_j}\frac{\partial M_p}{\partial x_k}, \tag{9d}$$

and where $\delta_{mnp}{}^{ijk} = e_{ijk} e_{mnp}$ is a generalized Kronecker symbol. By using (3d), (4), and (8c), the drag force density is

$$f_i^\alpha = d_{ij} v_j, \tag{10a}$$

the dissipation dyadic d_{ij} being defined by

$$d_{ij} = (-\alpha/M_s|\gamma|)(\partial M_k/\partial x_i)\partial M_k/\partial x_j. \tag{10b}$$

Finally, the reversible force is, from (1b) and (4),

$$f_i^r = (\delta\rho_E/\delta M_j)\, \partial M_j/\partial x_i. \tag{11}$$

Expressing the M_i in terms of M_s and the polar angles θ, φ and using vector notation converts

(9)–(11) into

$$\vec{f}^g = \vec{g} \times \vec{v}, \tag{12a}$$

$$\vec{g} = -(M_s/|\gamma|)\sin\theta(\nabla\theta)\times(\nabla\varphi), \tag{12b}$$

$$\vec{f}^{\alpha} = \overleftrightarrow{d}\cdot\vec{v}, \tag{13a}$$

$$\overleftrightarrow{d} = -(\alpha M_s/|\gamma|) \times[(\nabla\theta)(\nabla\theta)+\sin^2\theta(\nabla\varphi)(\nabla\varphi)], \tag{13b}$$

$$\vec{f}^r = (\delta\rho_E/\delta\theta)\nabla\theta + (\delta\rho_E/\delta\varphi)\nabla\varphi. \tag{14}$$

Note that g_i^2 is an invariant local measure of the extent to which the magnetic distribution depends on two spatial coordinates. The corresponding measure of the dependence of the magnetic distribution on three coordinates $\partial(M_1, M_2, M_3)/\partial(x_1, x_2, x_3) = 0$, since $M_i\,\partial M_i/\partial x_j = 0$.

The total domain reversible force and the total gyroscopic force integrals will now be carried out in general for steady-state motion. Since the spin configuration propagates in steady state by assumption, it is physically clear that only externally applied fields contribute to the reversible energy force. In order to consider this formally, it is convenient to divide the reversible force density into two terms,

$$\vec{f}^r = \vec{f}^{r\,\mathrm{in}} + \vec{f}^{r\,\mathrm{ex}}. \tag{15}$$

The internal force-density term $\vec{f}^{r\,\mathrm{in}}$ contains all forces due to anisotropy energy, exchange energy, internal demagnetizing fields, magnetostriction, etc. The external force density $\vec{f}^{r\,\mathrm{ex}}$ contains the force due to the externally applied field.

By using (14), the total internal reversible force is

$$\vec{F}^{r\,\mathrm{in}} = \int_V \left[\frac{\delta\rho_E^{\mathrm{in}}}{\delta\theta}(\nabla\theta) + \frac{\delta\rho_E^{\mathrm{in}}}{\delta\varphi}(\nabla\varphi)\right] dV. \tag{16a}$$

Since only variations corresponding to displacements are of interest, the $\delta\theta$ and $\delta\varphi$ at different field points are constrained to correspond to displacements so that

$$\vec{F}^{r\,\mathrm{in}} = -\int_V (\delta\rho_E^{\mathrm{in}}/\delta\vec{X})\,dV. \tag{16b}$$

The variation and integration are then interchanged with the result

$$\vec{F}^{r\,\mathrm{in}} = -\delta E_{\mathrm{in}}/\delta\vec{X} = 0 \tag{16c}$$

since the total energy E_{in} is invariant by assumption.

From (12) the total gyroscopic force is

$$\vec{F}^g = \int_V \vec{g}\times\vec{v}\,dV. \tag{17}$$

Since $\vec{v}$ is constant over the volume, (17) may be written as

$$\vec{F}^g = \vec{G}\times\vec{v}, \tag{18a}$$

$$\vec{G} \equiv \int_V \vec{g}\,dV, \tag{18b}$$

where $\vec{G}$ is the total gyrocoupling vector. From here on attention will be restricted to G_z, the expressions for G_x and G_y being entirely similar. Since the z component of (12b) may be written in terms of the Jacobian $\partial(\cos\theta, \varphi)/\partial(x, y)$ as

$$g_z = (M_s/|\gamma|)\partial(\cos\theta, \varphi)/\partial(x, y), \tag{19}$$

the z component of the total gyration is

$$G_z = \frac{M_s}{|\gamma|}\int_z\int_y\int_x \frac{\partial(\cos\theta, \varphi)}{\partial(x, y)}\,dx\,dy\,dz, \tag{20a}$$

$$= (M_s/|\gamma|)\int_z \Delta\cos\theta\,\Delta\varphi\,dz, \tag{20b}$$

and the transformation from (x, y) to $(\cos\theta, \varphi)$ is one-to-one so long as neither g_z nor g_z^{-1} is zero. In magnetic materials in which the exchange interaction is sufficiently strong, the exchange interaction prevents $g_z^{-1} = 0$. The surfaces (of whatever dimensionality), $g_z = 0$, form the boundaries of the regions over which the integration (20) is valid. In (20), $\Delta\cos\theta$ and $\Delta\varphi$ are thus the changes in $\cos\theta$ and φ from one $g_z = 0$ boundary to the next.

In the case of a cylindrical domain in a plate of thickness h oriented with the plate normal along the z axis, the domain wall separates a $g_z = 0$ line at or near the center of the domain from the $g_z = 0$ cylinder of infinite radius centered on the domain. In this case $\Delta\cos\theta = 2$ and $\Delta\varphi = 2\pi n$, where n is the integral number of times φ rotates about the z axis when the domain perimeter is traversed once in the direction of increasing φ. Although θ and φ may be functions of z, n must not be a function of z if the spin distribution is not to contain singularities. The total gyrocoupling vector of a cylindrical domain in an infinite plate of thickness h is thus

$$G_z = (4\pi M_s/|\gamma|)hn. \tag{21}$$

The steady-state motion of magnetic bubble domains is thus governed by the equation

$$\vec{F}^{r\,\mathrm{ex}} + (4\pi M_s/|\gamma|)hn\vec{i}_z\times\vec{v} + \vec{v}\cdot\int_V \overleftrightarrow{d}\,dV = 0, \tag{22}$$

where $\vec{F}^{r\,\mathrm{ex}}$ is the force due to the externally applied field. The integral of the dissipation dyadic is only weakly dependent on velocity at low velocity so that it may be estimated from the static spin distribution. It is well known that in any comparison with experiment the dissipative term must be modified by adding coercivity and/or making α depend on velocity. This tends to reduce the importance of any errors which arise in the estimation of the dissipation integrals since the coercivity term will appear in the same vector component in (22) as does the dissipation term.

As an example of the estimation of the dissipation dyadic, consider a cylindrical domain in which the Bloch lines are sufficiently dense so as to be in contact. It can be shown[7] that in a material whose energy density is

$$\rho_E = A[(\nabla\theta)^2 + \sin^2\theta(\nabla\varphi)^2] + K_u \sin^2\theta + 2\pi M_s^2 \sin^2\varphi \sin^2\theta \qquad (23)$$

there exists a planar wall solution to the corresponding Euler equations to first order in the parameter $q^{-1} = 2\pi M_s^2/K_u$ of the form $\cos\theta = \tanh[(\pi x/l_w)\varphi(y)]$, where $\varphi(y)$ is in general an elliptic function with period s. When the Bloch lines are in contact, the solution approaches $\varphi = \pm 2\pi y/s$ and $l_w^{-2} = K_u/\pi^{-2}A + (s/2)^{-2}$. The dissipation integral is evaluated for a section of this planar wall and then this result is applied to the cylindrical domain case neglecting curvature effects and assuming uniformly spaced Bloch line pairs, $s = \pi d/n$. Substituting this result in (22) and assuming a uniform applied field gradient ∇H_z, the velocity drive is

$$\tfrac{1}{8} d^2 |\gamma| \nabla H_z = n\vec{i}_z \times \vec{v} + \alpha n \frac{1 + 2N^2}{2N(1 + N^2)^{1/2}} \vec{v} = 0, \qquad (24)$$

where $N = 2nd^{-1}A^{1/2}K_u^{-1/2}$. When the Bloch line density becomes so large that $N > 1$, then the factor in (24) involving N rapidly approaches unity. Resolving (24) into components results in

$$|\nabla H_z| = |8nv/d^2\gamma|(1 + \alpha^2)^{1/2}, \qquad (25a)$$

$$\zeta = \tan^{-1}(1/\alpha), \qquad (25b)$$

where ζ is the angle between the velocity and the driving gradient. Slonczewski[6] has obtained similar expressions following a less-general approach while results identical to (25) have been obtained in a more specific calculation.[8]

Note in closing that, although the emphasis of this Letter is on the total domain forces, the force density expressions are useful in themselves as an aid in determing the internal structure of a moving domain. Even when the motion is not strictly steady state (such as motion driven by a thickness gradient), the instantaneous dissipation and gyroscopic forces may be used as first approximations.

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