cause the observed discrepancy. Uncertainties in the $K_{\mu 3}$ form factors also can contribute.

[6]Here we use the current world average for $\Gamma(K_L{}^0 \to \pi^+\pi^-)/\Gamma(K_L{}^0 \to \text{all})$ as obtained by the Particle Data Group, Lawrence Berkeley Laboratory Report No. 100, 1972 (unpublished).

Pion Condensation in Nuclear and Neutron Star Matter*

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The equilibrium thermodynamic conditions obeyed by a pion condensed system are given. These include the statement that the average spatial currents of conserved quantities must always vanish in the state of lowest free energy, even for π^- condensation into a running-wave mode. The physical significance of pion-condensation thresholds is also discussed.

The possibility that the pion field in dense nuclear or neutron-star matter may develop a macroscopically occupied mode, or condensate, has recently been explored by several authors.[1-4] In the normal state of these systems, the ground-state expectation values of φ_0, the neutral pion field, and φ, the charged (π^-) field, vanish as a result of parity conservation, and for $\langle\varphi\rangle$ as a result of charge conservation as well. Pion-condensed phases are states of broken symmetry, characterized by a nonvanishing $\langle\varphi\rangle$ for π^- condensation, while in a π^0 condensed system $\langle\varphi_0\rangle \neq 0$. The main purpose of this Letter is to derive the equilibrium thermodynamic conditions that are obeyed by the ground state, or, more generally at finite temperature, by the state of lowest free energy of a pion condensed system.

The first result we need, one familiar from the microscopic theory of superfluidity, is that the condensate wave function $\langle\varphi(\vec{r},t)\rangle$ must vary in time as $\exp(-i\mu_\pi t)$, where μ_π is the π^- chemical potential. To see this we note that in a π^- condensed phase, the ground-state energy $E = \langle H\rangle$, or the free energy $F = E - TS$ at finite temperature, is a functional of $\langle\varphi(\vec{r},t)\rangle$, $\langle\pi(\vec{r},t)\rangle$ (where π is the momentum conjugate to φ), as well as $\langle\varphi^\dagger(\vec{r},t)\rangle$ and $\langle\pi^\dagger(\vec{r},t)\rangle$. For fixed expectation value of $\rho_\pi = i(\varphi^\dagger\pi - \pi^\dagger\varphi)$, the ground-state energy $\langle H\rangle$ must be a minimum under variation of $\langle\varphi\rangle$ and $\langle\pi\rangle$, or, under an arbitrary variation, $\delta\langle H\rangle = \mu_\pi\,\delta\langle\rho_\pi\rangle$. Consider a variation that adds c-number fields to $\langle\varphi\rangle$ and $\langle\pi\rangle$. Then since $\delta\langle\rho_\pi\rangle = i\langle\varphi^\dagger\rangle\,\delta\langle\pi\rangle + \delta\langle\varphi^\dagger\rangle \times\langle\pi\rangle + \text{c.c.}$, and $\delta\langle H\rangle = \langle\delta H/\delta\pi\rangle\,\delta\langle\pi\rangle + \langle\delta H/\delta\varphi\rangle\,\delta\langle\varphi\rangle + \text{c.c.}$, we have, on comparing coefficients of $\delta\langle\varphi^\dagger\rangle$, and of $\delta\langle\pi^\dagger\rangle$,

$$\begin{aligned}\langle\delta H/\delta\pi^\dagger\rangle &\equiv \langle\dot{\varphi}\rangle = -i\mu_\pi\langle\varphi\rangle,\\ \langle\delta H/\delta\varphi^\dagger\rangle &\equiv -\langle\dot{\pi}\rangle = i\mu_\pi\langle\pi\rangle.\end{aligned} \tag{1}$$

Thus

$$\begin{aligned}\langle\varphi(\vec{r},t)\rangle &= \exp(-i\mu_\pi t)\langle\varphi(\vec{r})\rangle,\\ \langle\pi(\vec{r},t)\rangle &= \exp(-i\mu_\pi t)\langle\pi(\vec{r})\rangle.\end{aligned} \tag{2}$$

The π^0 chemical potential vanishes and so for a condensed neutral π^0 field $\langle\varphi_0\rangle$ and $\langle\pi_0\rangle$ are constant in time.

The density of condensed π^-, the net charge associated with the macroscopically occupied pion mode, is given by $\langle\rho_\pi\rangle_{\text{cond}} = -2\,\text{Im}(\langle\varphi\rangle^*\langle\pi\rangle)$. When terms in the interaction Lagrangian containing $\dot{\varphi}$ can be neglected, as in the nonrelativistic limit of the pseudovector coupling, then $\pi = \dot{\varphi}$ and[5] $\langle\rho_\pi\rangle_{\text{cond}} = 2\mu_\pi|\langle\varphi(\vec{r})\rangle|^2$.

Combining Eqs. (1) we find the expectation value of the π^- field equation:

$$(\mu_\pi{}^2 - m_\pi{}^2 + \nabla^2)\langle\varphi(\vec{r})\rangle - J(\vec{r}) = 0, \tag{3}$$

where $J(\vec{r}) = -\langle\delta L_{\text{int}}/\delta\varphi^\dagger(\vec{r})\rangle - i\mu_\pi\langle\delta L_{\text{int}}/\delta\dot{\varphi}^\dagger(\vec{r})\rangle$ is the source of the condensed-pion field and L_{int} is the interaction Lagrangian. Equation (3) determines the condensate wave function in terms of the source $J(\vec{r})$. The threshold for π^- condensation is the first point at which the field equation (3) can be satisfied; below threshold $J(\vec{r}) = 0$. Expanding $J(\vec{r})$ to first order in $\langle\varphi(\vec{r}\,')\rangle$ and using $\delta J(\vec{r})/\delta\langle\varphi(\vec{r}\,')\rangle = \Pi(\vec{r},\vec{r}\,';\omega=\mu_\pi)$, the π^- self-energy in the medium, we see that the π^--condensation threshold is the point where $\int D^{-1}(\vec{r},\vec{r}\,',\mu_\pi)\langle\varphi(\vec{r}\,')\rangle \times d^3r' = 0$, i.e., where the pion Green's function D, for $\langle\varphi\rangle = 0$, has a pole at frequency μ_π. Above the threshold, however, the amplitude of $\langle\varphi\rangle$ is determined by (3), not by the equation for D (which describes the fluctuations in the pion field). A point at which D_0, the π^0 Green's function (for $\langle\varphi_0\rangle = 0$), develops a pole at $\omega = \mu_{\pi 0} = 0$ would be a π^0 threshold.[1]

What is the physical significance of a pion condensation? It is instructive to compare it with the case of a hypothetical real scalar meson field σ coupled to the nucleon density. There the source of the σ field (the nucleon density) and thus $\langle\sigma\rangle$ is always nonzero; $\langle\sigma\rangle$ is essentially a nucleon self-energy field. There might occur a density however at which the σ Green's function develops a pole at $\omega=\mu_\sigma=0$. Such a pole would simply reflect a pole in the density-density correlation function of the nucleons at $\omega=0$ and would signal a phase transition of the matter analogous, if the pole is at $\vec{k}=0$, to a gas-liquid transition; if the pole occurs first at $\vec{k}\neq 0$ the transition would be to a state with a spatial nucleon density inhomogeneity. Such a transition could equally well be described without explicit introduction of the σ field as long as the interaction of the nucleons due to σ exchange is included, and retardation is neglected. There would be no new physics in "σ condensation."

The nonrelativistic source J_0 of the π^0 field is $\sim\nabla\cdot[\langle\Psi_p^\dagger\vec{\sigma}\Psi_p\rangle-\langle\Psi_n^\dagger\vec{\sigma}\Psi_n\rangle]$, the divergence of the isospin $T=1$ nucleon spin polarization. Even though the spin polarization can be nonzero, as in a nucleus with nonzero total spin, or in the presence of a magnetic field, its divergence, a pseudoscalar, must vanish in states of definite parity. A pole in $D_0(\omega=0)$ would reflect (for a nonrelativistic source) a corresponding pole in the longitudinal $T=1$ spin susceptibility of the matter at finite wave number; it would signal a phase transition to a state with a spatially varying spin order for which $\nabla\cdot\langle\sigma\rangle\neq 0$, a parity-nonconserving π^0 condensed phase. The order parameter $\langle\varphi_0\rangle$ of this phase is real. On the other hand a π^- condensed phase is physically very different since here the order parameter $\langle\varphi\rangle$ is generally complex. A phase in which $\langle\varphi\rangle\neq 0$ is superconducting, and is one in which there is an effective "pairing" of neutron particles with proton holes, and proton particles with neutron holes. (Note that the nonrelativistic source J of the π^- field is $\sim\nabla\cdot\langle\Psi_p^\dagger\vec{\sigma}\Psi_n\rangle$, which must be nonvanishing if $\langle\varphi\rangle\neq 0$.) The onset of π^- condensation in nuclear or neutron star matter would be indicated by the range of the normal state correlation function $\langle\Psi_p^\dagger(\vec{r}_1)\Psi_n(\vec{r}_1)\Psi_n^\dagger(\vec{r}_2)\Psi_p(\vec{r}_2)\rangle$ becoming infinite as a function of $|\vec{r}_1-\vec{r}_2|$.

We turn now to the thermodynamic conditions governing an equilibrium state of π^- condensation. Equilibrium under the reaction $n\leftrightarrow p+\pi^-$ implies that $\mu_\pi=\mu_n-\mu_p$, the difference of the n and p chemical potentials.[6] Quite generally, for any species i present in neutron star matter $\mu_i=b_i\mu_n-q_i\mu_e$, where μ_e is the electron chemical potential, b_i is the baryon number, and q_i the charge of species i. A more subtle question is what are the average velocities of the various components in the ground state. In a mixture of gases in which the number of each species is a constant of the motion, the spatial average of the ground-state expectation value of the particle current of each species must vanish. Sawyer and Scalapino,[2] in constructing their ground state for a π^--condensed system with $\langle\varphi\rangle\sim e^{i\vec{k}\cdot\vec{r}}$, imposed *ad hoc* the constraint that the average neutron momentum be zero, and that the average proton momentum be equal to and opposite the average pion momentum (computed as if the pions were a free field). Such a state has a nonvanishing baryon current. The correct condition, which is quite different, is that in a system in which the particles can change their identities, as in an e^-, n, p, π^- system, the spatial averages of the spatial (three) currents of the conserved quantities (here the baryon, charge, and two lepton currents) must vanish.

The proof follows from the fact that the ground-state energy must be invariant under an infinitesimal gauge transformation $\Psi_i(\vec{r})\to\exp[i\Lambda_i(\vec{r})]\times\Psi_i(\vec{r})$ of the complex fields, where for the boson fields, we transform φ_i and π_i by the same Λ_i in order to leave ρ_i invariant. Under such a transformation H is transformed into

$$H+\sum_i\int\vec{j}_i(\vec{r})\cdot\nabla\Lambda_i(\vec{r})\,d^3r,$$

plus terms of the forms, e.g.,

$$\int\left(\exp\{i[\Lambda_\pi(\vec{r})+\Lambda_p(\vec{r})-\Lambda_n(\vec{r})]\}-1\right)\mathcal{H}'(\vec{r})\,d^3r+\text{H.c.},\qquad(4)$$

where $\mathcal{H}'(\vec{r})$ is the sum of terms in the Hamiltonian density that transform π^-+p into n. Here $\vec{j}_i(\vec{r})$ is the current of species i. Let each of the Λ_i be of the form

$$\Lambda_i(\vec{r})=\sum_\alpha\alpha_i\Lambda_\alpha(\vec{r}),\qquad(5)$$

where the sum is over the four conserved quantities, indexed by $\alpha=b$ (baryon number), q (charge), l (electronlike lepton number), and l' (muonlike lepton number). [Thus for $\alpha=q$, α_i is the charge of the particles of species i.] The Λ_α are four independent functions. Then the conservation laws of baryon number, charge, and lepton numbers in the interaction Hamiltonian will guarantee that the terms of the form (4) in the transformed $\boldsymbol{H}$ will vanish. In this case

$$H \to H + \int d^3r \sum_\alpha \vec{j}_\alpha(\vec{r}) \cdot \nabla \Lambda_\alpha(\vec{r}),$$

where $\vec{j}_\alpha(\vec{r}) = \sum_i \alpha_i \vec{j}_i(\vec{r})$ are the four conserved currents. Since $\langle H\rangle$ must be a minimum under the gauge transformation (5) we must have, for each α,

$$\int d^3r \langle \vec{j}_\alpha(\vec{r})\rangle \cdot \nabla \Lambda_\alpha(\vec{r}) = 0. \qquad (6)$$

If we choose Λ_α of the form $\vec{a}\cdot\vec{r}$ (where $\vec{a}$ = const) then (6) implies that $\bar{\vec{j}}_\alpha$, the spatial average of $\langle \vec{j}_\alpha(\vec{r})\rangle$, vanishes. In a spatially uniform system the currents $\langle \vec{j}_b\rangle$, $\langle \vec{j}_q\rangle$, $\langle \vec{j}_l\rangle$, and $\langle \vec{j}_{l'}\rangle$ must vanish locally. Choosing the Λ's to vanish at infinity and integrating (6) by parts, we find in general that $\nabla\cdot\langle \vec{j}_\alpha(\vec{r})\rangle = 0$. When the Λ_i are arbitrary infinitesimal constants, the expectation values of terms of the form (4) must vanish; this implies that each $\int d^3r \langle \mathcal{H}'(\vec{r})\rangle$ is real, and also that the spatial average of $J(\vec{r})/\langle \varphi(\vec{r})\rangle$, Eq. (3), is real.

In applications to neutron star matter, one is concerned with states containing no neutrinos. The lepton currents are thus simply the electron and muon currents, and hence $\bar{\vec{j}}_e$ and $\bar{\vec{j}}$, the spatial averages of the electron and muon currents, must vanish. In a system containing only e^-, μ^-, n, and p under β equilibrium, the conditions $\bar{\vec{j}}_q = 0$ and $\bar{\vec{j}}_b = 0$ then imply $\bar{\vec{j}}_p = \bar{\vec{j}}_n = 0$. By contrast, in a system containing more degrees of freedom, say e^-, μ^-, n, p, and π^-, then all we can conclude from $\bar{\vec{j}}_\alpha = 0$ is that $\bar{\vec{j}}_{\pi^-} = \bar{\vec{j}}_p = -\bar{\vec{j}}_n$, but these $\bar{\vec{j}}$ need not vanish individually.[7]

We emphasize that the average electromagnetic current $\bar{\vec{j}}_q$ vanishes in the state of lowest energy, even for a running-wave π^- condensate $\langle\varphi\rangle \sim e^{i\vec{k}\cdot\vec{r}}$. For such a state, a variation in $\vec{k}$, at fixed $\bar{\vec{j}}_b = \bar{\vec{j}}_l = \bar{\vec{j}}_{l'} = 0$, produces a variation in the energy per unit volume $\delta\epsilon = \bar{\vec{j}}_q \cdot \delta\vec{k}$. This may be derived from the gauge-transformation arguments above. Thus choosing $\vec{k}$ to minimize ϵ is equivalent to setting $\bar{\vec{j}}_q = 0$. It should be noted that in the presence of velocity-dependent forces, such as the pion-nucleon pseudovector coupling, the pion current is not simply related to $\rho_\pi\vec{k}$; in fact in the calculation of Ref. 2 they should be oppositely directed. The general differential of the energy for the running-wave π^- case is

$$d\epsilon(\rho_i, |\langle\varphi\rangle|, \vec{k}) = \sum_i \mu_i\, d\rho_i + f d|\langle\varphi\rangle|^2 + \bar{\vec{j}}_q \cdot d\vec{k}, \qquad (7)$$

where $f = -\mu_\pi^2 + m_\pi^2 + k^2 + J/\langle\varphi\rangle$ is the left-hand side of the pion field equation. In equilibrium, $\sum_i \mu_i\, d\rho_i = \mu_n d\rho_b$. Condensates with $\vec{k}$ slightly different from its ground-state value correspond to supercurrent-carrying states.

Detailed model calculations will be reported later.[8] We mention here that in the simplest model of π^- condensation consisting of *noninteracting* n, p, and e^- with only a condensed-pion mode interacting with the nucleons, via a nonrelativistic pseudovector coupling, plus an s-wave π-N interaction, the condensation sets in at $\rho_b = 0.062$ fm^{-3}, corresponding to $\mu_n = 30.8$ MeV; k at threshold equals 1.1 fm^{-1}

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[5]This relation differs from the free-field relation employed in Refs. 2 and 3, between the pion field and $\langle\rho_\pi\rangle_{\rm cond}$. Equations (2) also imply that the value of the pion Hamiltonian is not simply $\langle\rho_\pi\rangle_{\rm cond}$ times the free pion energy.

[6]In symmetric nuclear matter without pion condensation $\mu_n = \mu_p$, and thus $\mu_\pi = 0$ at a π^--condensation threshold.

[7]Of course, the momentum of the state of lowest energy should vanish.

[8]G. Baym and E. Flowers, to be published.