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Method for Solving the Sine-Gordon Equation*

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The initial value problem for the sine-Gordon equation is solved by the inverse-scattering method.

The sine-Gordon equation,

$$\partial^2 u/\partial T^2 - \partial^2 u/\partial X^2 + \sin(u) = 0, \tag{1}$$

arises in many branches of mathematical physics.[1-3] Special solutions known as kinks,

$$u(X, T) = 4\tan^{-1}\{\exp[(X - UT)/(1-U^2)^{1/2}]\}, \tag{2}$$

have been known for some time and "multikink" solutions have also been found.[4-6]

Equation (1) can be solved by the inverse-scattering method.[7] It is convenient to write the sine-Gordon equation (1) as

$$\partial^2 u/\partial x\,\partial t = \sin(u), \tag{3}$$

by use of the transformation

$$x = \tfrac{1}{2}(X+T), \quad t = \tfrac{1}{2}(X-T). \tag{4}$$

Consider the following linear eigenvalue problem[8]:

$$\partial v_1/\partial x + i\zeta v_1 = q(x,t) v_2, \tag{5}$$

$$\partial v_2/\partial x - i\zeta v_2 = -q(x,t) v_1, \tag{6}$$

where ζ is the eigenvalue. $q(x,t) = -\frac{1}{2}\,\partial u(x,t)/\partial x$, and $u(x,t) \to 0$ (or a multiple of 2π) as $x \to \pm\infty$ and is sufficiently well behaved. By cross differentiation, a particular choice for the t dependence of $v_1(x,t)$ and $v_2(x,t)$,

$$\partial v_1/\partial t = (i/4\zeta)(v_1 \cos u + v_2 \sin u), \tag{7}$$

$$\partial v_2/\partial t = (i/4\zeta)(v_1 \sin u - v_2 \cos u), \tag{8}$$

ensures that the eigenvalue ζ is independent of t, provided $u(x,t)$ evolves according to (3). The scattering data (eigenvalues, reflection coefficients, etc.) can be found and the potential $q(x,t)$ can then be determined for all time by the inverse-scattering method.

The results of Zakharov and Shabat[8] directly apply and allow us to reconstruct $q(x,t)$!

Following Ref. 8, we define the functions φ and ψ as solutions of Eqs. (5) and (6) with the asymptotic form

$$\begin{aligned} \varphi &\to \begin{bmatrix}1\\0\end{bmatrix} e^{-i\zeta x} \text{ as } x \to -\infty, \quad \operatorname{Im}\zeta \geq 0; \\ \psi &\to \begin{bmatrix}0\\1\end{bmatrix} e^{i\zeta x} \text{ as } x \to +\infty, \quad \operatorname{Im}\zeta \geq 0. \end{aligned} \tag{9}$$

In addition, we note that if

$$\psi(x,\zeta) = \begin{bmatrix}\psi_1(x,\zeta)\\ \psi_2(x,\zeta)\end{bmatrix}$$

is a solution of (5) and (6), then a linearly independent solution is given by

$$\overline{\psi}(x,\zeta) = \begin{bmatrix}\psi_2^*(x,\zeta^*)\\ -\psi_1^*(x,\zeta^*)\end{bmatrix}. \tag{10}$$

The pair of solutions ψ and $\overline{\psi}$ forms a complete system of solutions, and therefore we can write

$$\varphi = a(\xi)\overline{\psi} + b(\xi)\psi, \quad \zeta = \xi + i\eta. \tag{11}$$

$a(\xi)$ can be analytically continued to the upper half-plane $\operatorname{Im}(g\zeta) > 0$ and, in particular, the zeros ζ_j ($j = 1, \ldots, N$) of $a(\zeta)$ in the upper half-plane are the discrete eigenvalues of (5) and (6). At these values we have

$$\varphi(x,\zeta_j) = c_j \psi(x,\zeta_j). \tag{12}$$

The t dependence of $a(\xi)$, $b(\xi)$, and c_j is found from (7) and (8) to be

$$\begin{aligned} &a(\xi) = a_0(\xi), \quad b(\xi) = b_0(\xi)\exp(-it/2\xi), \\ &c_j(\zeta_j) = c_{j0}\exp(-it/2\zeta_j). \end{aligned} \tag{13}$$

The solution of the inverse problem[8] is given by the integral equation

$$K(x, y) = B^*(x+y) - \int_x^\infty \int_x^\infty B^*(y+z)B(k+z)K(x, k)\,dz\,dk, \tag{14}$$

where

$$B(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{b(\xi)}{a(\xi)} \exp(i\xi x)\,d\xi - i\sum_{j=1}^{N} c_j \exp(i\zeta_j x). \tag{15}$$

In this case, since u is real it follows directly that $B(x)$ is real. The potential $q(x, t)$ is given by

$$q(x, t) = -\tfrac{1}{2}\,\partial u(x, t)/\partial x = -2K(x, x), \tag{16}$$

and the solution

$$u(x, t) = u(t, t) + 4\int_t^x K(y, y; t)\,dy. \tag{17}$$

Now let us consider how to determine the scattering data from the initial-value ($T=0$) problem. Rewriting Eqs. (5)–(8) in real space and time (X, T) variables, we have

$$\frac{\partial v_1}{\partial X} = \left(-\frac{i\zeta}{2} + \frac{i}{8\zeta}\cos u\right)v_1 + \left[\frac{i}{8\zeta}\sin u - \frac{1}{4}\left(\frac{\partial u}{\partial X} + \frac{\partial u}{\partial T}\right)\right]v_2, \tag{18}$$

$$\frac{\partial v_2}{\partial X} = \left[\frac{i}{8\zeta}\sin u + \frac{1}{4}\left(\frac{\partial u}{\partial X} + \frac{\partial u}{\partial T}\right)\right]v_1 + \left(\frac{i\zeta}{2} - \frac{i}{8\zeta}\cos u\right)v_2, \tag{19}$$

$$\frac{\partial v_1}{\partial T} = \left(-\frac{i\zeta}{2} - \frac{i}{8\zeta}\cos u\right)v_1 - \left[\frac{i}{8\zeta}\sin u + \frac{1}{4}\left(\frac{\partial u}{\partial X} + \frac{\partial u}{\partial T}\right)\right]v_2, \tag{20}$$

$$\frac{\partial v_2}{\partial T} = \left[-\frac{i}{8\zeta}\sin u + \frac{1}{4}\left(\frac{\partial u}{\partial X} + \frac{\partial u}{\partial T}\right)\right]v_1 + \left(\frac{i\zeta}{2} + \frac{i}{8\zeta}\cos u\right)v_2. \tag{21}$$

Again it can be directly verified that $\partial\zeta/\partial T = \partial\zeta/\partial X = 0$. Define the functions Φ and and Ψ which approach the following asymptotic values. At any time T,

$$\begin{aligned}\Phi &\to \begin{bmatrix}1\\0\end{bmatrix}\exp\left[-\frac{i}{2}\left(\zeta - \frac{1}{4\zeta}\right)X - \frac{i}{2}\left(\zeta + \frac{1}{4\zeta}\right)T\right] \text{ as } X \to -\infty, \quad \operatorname{Im}\zeta \geq 0,\\ \Psi &\to \begin{bmatrix}0\\1\end{bmatrix}\exp\left[\frac{i}{2}\left(\zeta - \frac{1}{4\zeta}\right)X + \frac{i}{2}\left(\zeta + \frac{1}{4\zeta}\right)T\right] \text{ as } X \to +\infty, \quad \operatorname{Im}\zeta \geq 0;\end{aligned} \tag{22}$$

and define

$$\overline{\Psi}(X, \zeta) = \begin{bmatrix}\Psi_2^*(X, \zeta^*)\\ -\Psi_1^*(X, \zeta^*)\end{bmatrix}. \tag{23}$$

Since Ψ and $\overline{\Psi}$ are linearly independent, we may write

$$\Phi(X, \xi) = A_0(\xi)\overline{\Psi}(X, \xi) + B_0(\xi)\Psi(X, \xi). \tag{24}$$

Again $A_0(\zeta)$ can be analytically continued into the upper half-plane, and the zeros of $A_0(\zeta)$ are the eigenvalues of (18) and (19) [and therefore also of (5) and (6)]. At the eigenvalues ζ_j, $j = 1, \ldots, N$,

$$\Phi(X, \zeta_j) = C_{j0}\Psi(X, \zeta_j). \tag{25}$$

By using (20) and (21) it can readily be verified that $A_0(\xi)$, $B_0(\xi)$, and C_{j0} are independent of T. These are known quantities and can be related to the unknown qunatities in Eq. (13) by

$$a_0(\zeta) = A_0(\zeta), \quad b_0(\xi) = B_0(\xi), \quad c_{j0} = C_{j0}. \tag{26}$$

It should be remarked that the discrete eigenvalues must be either purely imaginary or arise as complex conjugate pairs: $\zeta, -\zeta^*$. Corresponding to one purely imaginary eigenvalue, $\zeta = i\eta$ is a special traveling-wave solution of the form

$$u = 4\tan^{-1}(e^{\pm\theta}), \quad \theta = (\eta + 1/4\eta)(X - X_0) + (\eta - 1/4\eta)T. \tag{27}$$

Similarly, paired complex eigenvalues correspond to soliton states which oscillate in time ("breathers"). A particular breather solution with $\xi\xi^* = \frac{1}{4}$ is given by

$$u = 4\tan^{-1}\{[(1-\omega^2)/\omega^2]^{1/2}\cos[\omega(T-T_0)]\,\mathrm{sech}[(1-\omega^2)^{1/2}(X-X_0)]\}, \quad (28)$$

where $\omega = -2\,\mathrm{Re}\,\xi$. We note that the eigenvalues corresponding to the modes of a given breather in its own rest frame lie on the circle $\xi\xi^* = \frac{1}{4}$.

When $b(\xi) = 0$ the solution is generated by the discrete spectrum only. Following Ref. 8, the solution can be shown to be given by

$$\frac{1}{4}\left(\frac{\partial u}{\partial x}\right)^2 = \frac{d^2}{dx^2}\ln[\det(I + AA^*)], \quad (29)$$

where

$$A_{lm} = \frac{(c_l c_m^*)^{1/2}}{\zeta_l - \zeta_m^*}\exp[i(\zeta_l - \zeta_m^*)X]. \quad (30)$$

The phase shifts of kink-kink (kink-antikink) interactions have been discussed already in the literature.[6] Lamb[9] has recently investigated a special class of solutions which correspond to paired complex eigenvalues.

A method for generating an infinite set of conservation laws has been given by Lamb[10] and applied in the context of nonlinear optics.[11]

In general the solution depends on both the discrete and continuous spectrum $[b(\xi) \neq 0]$. The asymptotic behavior of u in these cases can be found by using methods similar to those used by Ablowitz and Newell.[12] That part of the solution corresponding to the continuous spectrum decays algebraically in time. In this regard it should be noted that, corresponding to initial conditions of compact support, the continuous spectrum completely cancels any contribution arising from the discrete spectrum outside the light cone at any $T \neq 0$.

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Cosmic $\bar{p}$ Production in Interstellar pp Collisions

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We place a conservative upper limit of $\sim 4.6\times10^{-4}$ on the fraction of antiprotons expected in the cosmic-ray flux at the top of Earth's atmosphere due to collisions of primary cosmic rays with interstellar hydrogen. The implication of this result for experiments designed to detect the existence of antimatter in the universe is discussed. We have parametrized existing data on the $\bar{p}$ inclusive cross section from threshold to intersecting-storage-ring energies to obtain the result.

Recent experiments designed to look for antimatter in the cosmic radiation have concentrated on antihelium[1,2] and heavier antinuclei[1] in order to be able to neglect significant contamination of the flux of antinuclei by antiprotons produced in the collision of ordinary cosmic rays with the interstellar gas. Previous calculations of the $\bar{p}$ flux to be expected from this source have been