ture of multiple scattering is dependent upon the mathematical formalism used.

At 1 GeV, $\rho = -0.2$ is an average of proton-proton and neutron-proton parameters, both parameters obtained from two independent nucleon-nucleon determinations[9,10] which lead to the same results. It can thus be presumed to be fixed by n-n data and independent of n-N experiments. The value $\rho = -0.3$, used in Glauber-type fits to n-N data,[2,4] was determined via a best fit to the larger-angle p-^{4}He data. It thus really represents the ratio of the real to imaginary parts of a phenomenological potential (e.g., see line 2, Table I of Ref. 6) and, as such, includes nuclear structure and recoil effects; it cannot be assumed to be a good representation for the free, forward, nucleon-nucleon amplitude.

The fact that the "correct" value of ρ, when used in the single-scattering potential, leads to a good representation of the small-angle p-^{4}He data indicates that the latter approach can be used to determine ρ at other energies where the n-n results are not conclusive. At 0.6 GeV, there is a large discrepancy in the n-n determination of ρ: Bugg *et al.*[9] imply a value of $+0.145$, whereas the results of Dutton *et al.*[10] lead to $\rho \sim -0.7$. Boschitz *et al.*,[11] using a Glauber approach, determine $\rho = -0.43$ by fitting the data for p-^{4}He at the diffraction minimum, but this value suffers from the disadvantages mentioned above: It is an n-N value, not an n-n parameter. Using ρ as the only free parameter in our approach, we find an excellent fit to the 0.6-GeV data, for p-^{4}He center-of-mass angles less than 16° with $\rho = -0.14$, a value much closer to the n-n value at 1 GeV. A direct, independent n-n determination of this parameter at 0.6 GeV—as was done in Ref. 10 at 1 GeV—would be a useful check on the kinematic formalism and results proposed in this paper.

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Fusion Barriers in Heavy-Ion Reactions

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We present experimental fusion barriers for ^{32}S ions on ^{24}Mg, ^{27}Al, ^{40}Ca, and ^{58}Ni. These and published data for Ar and Kr ion-induced reactions are analyzed in terms of a simple classical formula for barrier heights. A prescription based on equivalent uniform charge radii from electron scattering is shown to reproduce all results to within the experimental uncertainty.

There has been much speculation, and there have been many calculations concerning the effective barrier for reactions between heavy ions and target nuclei.[1-4] There have been predictions of Coulomb-induced distortion during approach of the ions, and quantitative estimates of the Z dependence of such distortion effects. An interesting model prediction made more recently by Wong suggests that the minimum radius for interaction decreases with increasing target or projectile charge with respect to predictions of a classical barrier formula with a radius parameter r_0 which is mass independent.[4] In this Letter we will present new experimental fusion barrier results for ^{32}S ion-induced reactions on ^{24}Mg, ^{27}Al, ^{40}Ca, and ^{58}Ni targets. These will be analyzed along with published results for Ar and Kr ion-induced reactions on targets up to ^{238}U. An attempt will be made to analyze these results in a fairly model-independent manner based on radii resulting from electron scattering and muonic-atom experiments, and it will be shown that a simple barrier parametrization results for all systems to within the ±2–3% uncertainty of the experimental results.

Fusion-cross-section measurements were made using ^{32}S beams from the Rochester MP tandem Van de Graaff accelerator. Gas proportional $\Delta E/E$ solid-state counter telescopes were used for direct measurement of fusion product angular distributions. The angular distributions were measured in 1°–2° steps (laboratory system) at angles between 3° and 20°; these results were then integrated over angle to give total fusion product cross sections. Results, as shown in Fig. 1, are thought to be accurate to ±10%. These results were then plotted against the reciprocal of the center-of-mass energy to determine the fusion barriers V. This analysis is based on the classical formula

$$\sigma = \pi R^2(1 - V/\epsilon). \tag{1}$$

The linearly extrapolated zero-cross-section intercepts gave the barriers V with an estimated uncertainty of ±2% (slightly higher for ^{32}S + ^{58}Ni). The distance of the onset of fusion at $\epsilon = V$ is R_{fus}, and can be evaluated to ±0.2 fm from the slope of the σ-versus-$1/\epsilon$ line. Results for R_{fus} and V from these data are given in Table I along with experimental barrier heights which have been published for Ar- and Kr-induced reactions.[5-9]

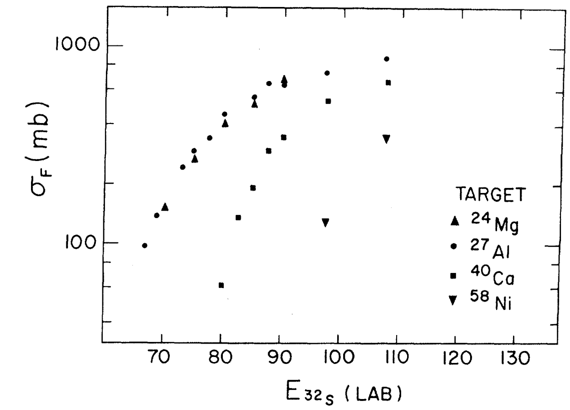

FIG. 1. Experimental fusion cross section as a function of incident ^{32}S ion energy (MeV). Target nuclides are indicated on the figure.

The interaction potential between two ions may be represented as the sum of repulsive Coulomb and attractive nuclear potentials,

$$V(R) = V_{\text{Coul}}(R) + V_{\text{nucl}}(R) \tag{2}$$

and the fusion barrier V is defined to be the value of $V(R)$ at which $dV(R)/dR = 0$. The fusion radius, R_{fus} of Eq. (1), is the value of R where the derivative is zero. Values of fusion barriers as defined by Eq. (2) and as deduced from experimental results have often been analyzed by a parametrization based only on the Coulomb interaction term,[9,10]

$$V = Z_1Z_2e^2/R_{\text{eff}} = Z_1Z_2e^2/(R_{HI} + R_T + d), \tag{3}$$

TABLE I. Experimental fusion barrier heights and radii for fusion reactions.

Projectile	Target	$V_{\text{c.m.}}$ (MeV)	R_{fus} (fm)	R_{eff} (fm)	$R_{\text{eq. uniform}}$ (fm)	d (fm)	Source
^{32}S	^{24}Mg	28.3	8.7 ± 0.2	9.8	8.0	1.8	This work
	^{27}Al	29.7	8.5 ± 0.2	10.1	8.1	2.0	This work
	^{40}Ca	43.5	9.2 ± 0.2	10.6	8.7	1.9	This work
	^{58}Ni	59.5	8.5 ± 0.25	10.85	9.1	1.8	This work
^{40}Ar	^{169}Dy	135		12.8	11.2	1.6	Ref. 5
	^{238}U	171		13.9	11.9	2.0	Ref. 6
^{84}Kr	^{72}Ge	145		11.4	10.2	1.2	Ref. 7
	^{116}Cd	204		12.3	11.1	1.2	Ref. 7
	^{232}Th	332		14.1	12.5	1.6	Ref. 8
	^{238}U	333		14.3	12.5	1.8	Ref. 8

where R_{HI} and R_T are heavy-ion and target radii, and d is an additional distance parameter which allows for effects such as finite range of nuclear forces, differences between proton and neutron distributions, diffuseness, and possible distortion effects. In particular it should be noted that the distance R_{eff} extracted from Eq. (3) upon substitution of experimental fusion barriers V, must be larger than the values of R_{fus} of Eq. (1). This follows from the fact that $V_{nucl}(R)$ has been neglected in assuming $V(R) = V_{Coul}(R)$ in the parametrization of Eq. (3). An analysis of the type implied by Eq. (3), in which experimental results are compared with calculated values of $R_{HI} + R_T$, and d values are extracted, has been reported by Lefort *et al.*[9] A constant radius parameter was used in calculating R_{HI} and R_T in Ref. 9. We have analyzed the data summarized in Table I using values of R_{HI} and R_T from equivalent uniform charge radii which were experimentally determined in electron scattering and mesonic-atom experiments[10,11]; in this respect the analysis differs from several other treatments of the problem in which a mass-independent radius parameter was chosen.[4,9] For nuclei not listed in the tabulations of Ref. 11 or by Überall,[12] equivalent uniform charge radii based on radius parameters extracted for the nearest listed nuclei were used. This may give the largest uncertainty for reactions on the deformed Th and U targets, for which a radius parameter based on the spherical nuclei ^{208}Pb and ^{209}Bi was used.

The values extracted for d are summarized in Table I, and in Fig. 2. Within the experimental uncertainties of $\pm 0.2 - 0.3$ fm, all results are consistent with an interaction radius of $R_{HI} + R_T + 1.7$ fm. Trends toward lower radii for higher target or projectile charge (lower d values for higher Z) are not supported by the present data when the radii are evaluated from experimental results. This point is emphasized in the upper portion of Fig. 2 where the d values are shown as ordinate with $Z_{HI} Z_T$ as abscissa.

The equivalent uniform charge electron scattering radii are reproduced to within 0.1 fm by the analytical result due to Elton,[12]

$$R = 1.120A^{1/3} + 2.009A^{-1/3} - 1.513A^{-1} \text{ fm}. \qquad (4)$$

The results of the analyses of this work would therefore suggest that reaction barriers for quite heavy ions might be estimated to 2–3% using Eqs. (3) and (4) with $d \cong 1.7$ fm.

We conclude that to within uncertainties of ±2–3%, the data analyzed herein present no experimental evidence for deformation or charge effects on fusion barrier heights when the data are analyzed by the method presented in this work. Our interpretation of the conclusions of Refs. 1–3 is that no effects outside this range are predicted. More extensive data will be needed to answer the question as to whether or not such effects are present and of quantitative importance with respect to limitations of experimental determination and significance, or whether they would be present if a radius based on the mass rather than the charge distribution were used. In the interim, the prescription for calculating reaction barriers given by Eqs. (3) and (4) may be useful.

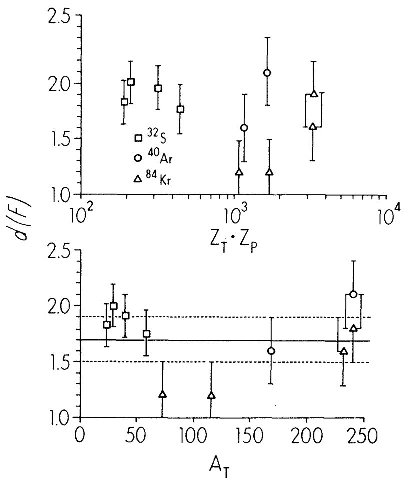

FIG. 2. Distance parameters extracted from the barrier-height formula. Upper part, d as a function of the product of target and projectile charge; lower part, d as a function of target mass. A solid curve has been drawn at $d = 1.7$ fm, and dotted curves at 1.7 ± 0.2 fm to illustrate that d may be considered a constant to within those limits. The projectiles as identified in the upper curve correspond to points in both curves.

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Method for Solving the Sine-Gordon Equation*

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The initial value problem for the sine-Gordon equation is solved by the inverse-scattering method.

The sine-Gordon equation,

$$\partial^2 u/\partial T^2 - \partial^2 u/\partial X^2 + \sin(u) = 0, \quad (1)$$

arises in many branches of mathematical physics.[1-3] Special solutions known as kinks,

$$u(X, T) = 4\tan^{-1}\{\exp[(X - UT)/(1-U^2)^{1/2}]\}, \quad (2)$$

have been known for some time and "multikink" solutions have also been found.[4-6]

Equation (1) can be solved by the inverse-scattering method.[7] It is convenient to write the sine-Gordon equation (1) as

$$\partial^2 u/\partial x\,\partial t = \sin(u), \quad (3)$$

by use of the transformation

$$x = \tfrac{1}{2}(X + T), \quad t = \tfrac{1}{2}(X - T). \quad (4)$$

Consider the following linear eigenvalue problem[8]:

$$\partial v_1/\partial x + i\zeta v_1 = q(x, t) v_2, \quad (5)$$

$$\partial v_2/\partial x - i\zeta v_2 = -q(x, t) v_1, \quad (6)$$

where ζ is the eigenvalue. $q(x, t) = -\frac{1}{2}\,\partial u(x, t)/\partial x$, and $u(x, t) \to 0$ (or a multiple of 2π) as $x \to \pm\infty$ and is sufficiently well behaved. By cross differentiation, a particular choice for the t dependence of $v_1(x, t)$ and $v_2(x, t)$,

$$\partial v_1/\partial t = (i/4\zeta)(v_1 \cos u + v_2 \sin u), \quad (7)$$

$$\partial v_2/\partial t = (i/4\zeta)(v_1 \sin u - v_2 \cos u), \quad (8)$$

ensures that the eigenvalue ζ is independent of t, provided $u(x, t)$ evolves according to (3). The scattering data (eigenvalues, reflection coefficients, etc.) can be found and the potential $q(x, t)$ can then be determined for all time by the inverse-scattering method.

The results of Zakharov and Shabat[8] directly apply and allow us to reconstruct $q(x, t)$!

Following Ref. 8, we define the functions φ and ψ as solutions of Eqs. (5) and (6) with the asymptotic form

$$\varphi \to \begin{bmatrix}1\\0\end{bmatrix} e^{-i\zeta x} \text{ as } x \to -\infty, \quad \mathrm{Im}\zeta \geq 0;$$
$$\psi \to \begin{bmatrix}0\\1\end{bmatrix} e^{i\zeta x} \text{ as } x \to +\infty, \quad \mathrm{Im}\zeta \geq 0. \quad (9)$$

In addition, we note that if

$$\psi(x, \zeta) = \begin{bmatrix}\psi_1(x, \zeta)\\ \psi_2(x, \zeta)\end{bmatrix}$$

is a solution of (5) and (6), then a linearly independent solution is given by

$$\bar{\psi}(x, \zeta) = \begin{bmatrix}\psi_2^*(x, \zeta^*)\\ -\psi_1^*(x, \zeta^*)\end{bmatrix}. \quad (10)$$

The pair of solutions ψ and $\bar{\psi}$ forms a complete system of solutions, and therefore we can write

$$\varphi = a(\xi)\bar{\psi} + b(\xi)\psi, \quad \zeta = \xi + i\eta. \quad (11)$$

$a(\xi)$ can be analytically continued to the upper half-plane $\mathrm{Im}(g\zeta) > 0$ and, in particular, the zeros ζ_j ($j = 1, \ldots, N$) of $a(\zeta)$ in the upper half-plane are the discrete eigenvalues of (5) and (6). At these values we have

$$\varphi(x, \zeta_j) = c_j \psi(x, \zeta_j). \quad (12)$$

The t dependence of $a(\zeta)$, $b(\xi)$, and c_j is found from (7) and (8) to be

$$a(\zeta) = a_0(\zeta), \quad b(\xi) = b_0(\xi)\exp(-it/2\xi),$$
$$c_j(\zeta_j) = c_{j0}\exp(-it/2\zeta_j). \quad (13)$$

The solution of the inverse problem[8] is given