particle transfer reactions. For the $^{240}$Pu final nucleus, the ratio of the strengths for population of the excited $K^\pi = 0^+$ bands to those for the ground-state bands is the same for the reaction $^{239}$Pu$(d,p)$ as for $^{242}$Pu$(p,t)$. For the $^{238}$Pu final nucleus, the corresponding ratio for the reaction $^{239}$Pu$(d,t)$ is only $\frac{1}{4}$ of that for the reaction $^{240}$Pu$(p,t)$.

Since the relative population of the $K^\pi = 0^+$ states is as large for the $(d,p)$ reaction as it is for the $(p,t)$, and since the relative population in the $(d,t)$ reaction is still a quarter of that in the $(p,t)$, it appears that the prediction of van Rij and Kahana[7] is not upheld in these two cases. However, it should be noted that we have not taken two-step processes into account, and these could be responsible for the strengths seen in both reactions.

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# Qualitative Theory of Pion Scattering by Nuclei*

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The requirement of unitarity, together with the theory of the refractive index, puts a limit on the wave number of a pion (of given energy) in nuclear matter (of given density). A paradox in the conventional theory of pion scattering by complex nuclei is thereby avoided.

The standard theory of the scattering of pions by complex nuclei is based on the Kisslinger potential.[1] Several numerical calculations have been carried out[2-4] with this potential, and agreement with experiment has generally been satisfactory. There is also an extensive literature on modifications of this potential and other approaches.

The Kisslinger potential has, however, a very puzzling feature, as follows. Using Ref. 2a, Eqs. (20) to (22), and neglecting the Coulomb potential, we find for the wave number of a pion in nuclear matter of *constant* density[5] $\rho$

$$k^2 = k_0^2(1 + b_0\rho)/(1 - b_1\rho), \tag{1}$$

where $k_0$ is the free-space wave number, $\rho$ the density of nuclear matter, $k_0^2 = \omega^2 - \mu^2$, and $b_0$ and $b_1$ are parameters describing the $s$ and $p$ scattering of a pion by a nucleon. These are determined by Auerbach *et al.* from the scattering of $\pi$ by *free* nucleons, with the result (at 80-MeV lab energy)

$$b_1 = 6.5 + 1.8i \text{ F}^3. \tag{2}$$

Disregarding the imaginary part, $k^2$ becomes infinite at $\rho = 0.16$ F$^{-3}$, i.e., just about the density of normal nuclear matter, and becomes negative for higher density. This is a quite unphysical behavior.

The paradox is solved by two considerations, viz. (a) unitarity and (b) the Pauli principle. To discuss this problem, we start from the general theory of the refractive index which gives

$$k^2 - k_0^2 = 4\pi\rho f(k, 0), \tag{3}$$

where $f(k, 0)$ is the forward scattering amplitude of a particle of wave number $k$. This equation was derived by Goldberger and Seitz[6] from the Schrödinger equation, but it follows in fact directly from Huyghens principle regardless of the wave equation.[7] Now the *maximum* possible value of $f$, when only one partial wave $l, j$ contributes, is

$$|f_{\mathrm{unit}}| = (j + \tfrac{1}{2})/k \tag{4}$$

(subscript unit for unitary). In our case, the relevant state is $p_{3/2}$ (isospin will be considered later), so $j + \frac{1}{2} = 2$, and

$$|k^2 - k_0^2| \leq 8\pi\rho/k. \tag{5}$$

Here $k$ occurs because only the "wave in the medium" is incident on the scatterer; the "free" wave $k_0$ is extinguished upon entry into the medium. Assuming $\rho$ large and $k_0$ not very large, we find

$$|k^3| < 8\pi\rho, \tag{5a}$$

so the infinite result of the Kisslinger theory can never occur.

In accord with general practice, we set

$$f(k, 0) = 2\sin\delta e^{i\delta}/k, \tag{6}$$

where $\delta$ refers to the 33 state. The Chew-Low theory[8] gives

$$\frac{\tan\delta}{k^3} = \frac{4}{3}\frac{f^2}{\mu^2}\frac{1}{\omega(1 - \omega/\omega_r)}, \tag{7}$$

where $\omega_r$ is the resonance energy, 298 MeV in the c.m. system,[9] corresponding to $\omega_r/\mu = 2.14$, and $k_r/\mu = 1.89$. Further, $f^2 = 0.088$, $\omega$ is the energy, and $k$ is again the wave number *in the medium.* This comes in because the pseudovector coupling is $\sigma \cdot \nabla\varphi \approx \vec{\sigma} \cdot \vec{k}\varphi$, where $\varphi$ is the pion wave function, and this is the origin of two powers of $k$ on the left-hand side of (7). Chew and Low clearly separate, in their Eq. (33), the simple dependence of the scattered amplitude on $k$ from the more intricate dependence on $\omega$. The latter is mainly caused by intermediate pion states of high momentum, as is shown by their Eq. (49), and these are likely to be about the same whether the nucleon is free or in nuclear matter. Therefore the form of the $\omega$ dependence, their Eq. (51), is taken over into our theory, including the value of the resonance energy. The relation of the center of mass to the lab energy of the $\pi$, in view of the Fermi motion of the nucleons, is discussed, e.g., by Landau, Phatak, and Tabakin.[10]

For algebraic simplicity, we now make the following approximations:

$$\begin{aligned} &\sin\delta e^{i\delta} = \tan\delta \quad \text{if } \tan\delta < 1, \\ &\sin\delta e^{i\delta} = 1 \text{ if } \tan\delta > 1, \end{aligned} \tag{8}$$

$\tan\delta$ being calculated from (7). The complex phase $e^{i\delta}$ will be calculated later, using inelastic scattering. If the first Eq. (8) and (6) are inserted into (3), we get the Kisslinger approximation. But we then have to ascertain whether (7) gives $\tan\delta < 1$, and this is generally not the case if $k$ becomes very large, as it does in the Kisslinger paradox.

For $\pi^-$, (6) is the correct amplitude for scattering by neutrons; for protons the amplitude is $\frac{1}{3}$ of this, so that

$$\rho = \rho_n + \tfrac{1}{3}\rho_p \equiv \kappa\rho_n. \tag{9}$$

We write $\rho_n$ in terms of the Fermi momentum:

$$\rho_n = p_F^3/3\pi^2. \tag{10}$$

For normal nuclear matter, $p_F = 1.34\ \mathrm{F}^{-1} = 1.89\mu$. In the simple case when $\tan\delta > 1$, we have then from (3), (6), and (8)

$$k^2 - k_0^2 = (8\kappa/3\pi)p_F^3/k. \tag{11}$$

If $k_0 < p_F$, $k$ will be of the order of $p_F$, i.e., neither of the order of $k_0$ nor extremely large.

Since $k$ is of order $p_F$, the *Pauli principle* will have an effect. Figure 1 shows the mechanism for the 33 scattering of a $\pi^-$ by a neutron of initial momentum $p$: The final $\pi^-$ of momentum $\vec{k}'$ is emitted, and the neutron is thereby transformed into a proton of momentum $\vec{p} - \vec{k}'$; then the initial $\pi^-$ is absorbed and a neutron of momentum $\vec{p} - \vec{k}' + \vec{k}$ results. In forward scattering, $\vec{k}' = \vec{k}$ and the final neutron state is the same as the initial. The process, however, is forbidden if the intermediate proton state $\vec{p} - \vec{k}$ is occupied. Assuming equal density of protons and neutrons, the *fraction* of initial neutron states for which the inter-

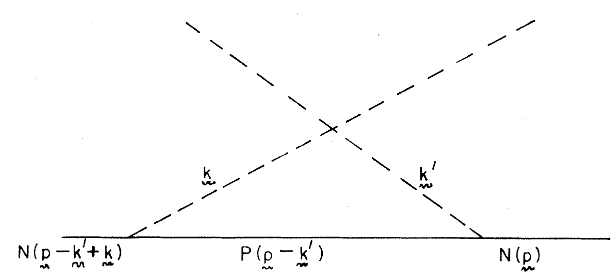


FIG. 1. Feynman diagram for 33-resonance scattering.

mediate state $p-k$ is empty is

$$F=\tfrac{3}{4}(k/p_F)(1-\tfrac{1}{12}k^2/p_F^2) \text{ if } k<2p_F,$$
$$F=1 \text{ if } k>2p_F. \tag{12}$$

For small $k$, and even for $k$ near $p_F$, it is a sufficient approximation to set

$$F=3k/4p_F. \tag{12a}$$

We simply multiply $\rho$ by this factor, saying that only the fraction $F$ of the neutrons is able to scatter $\pi^-$. Inserting $\kappa=\frac{4}{3}$, Eq. (11) becomes then

$$k^2-k_0^2=\frac{8p_F^2}{3\pi}. \tag{13}$$

As previously pointed out, this is only valid if $\tan\delta>1$. It is the simplest dispersion relation conceivable.

If $\tan\delta<1$, we use Eqs. (6)–(10) and (12a) in (3) and get

$$k^2-k_0^2=\frac{8}{3\pi}p_F^2\frac{4}{3}\frac{f^2}{\mu^2}\frac{k^3}{\omega(1-\omega/\omega_r)}. \tag{14}$$

The ratio of (14) to (13) represents $\tan\delta$; therefore, in accord with (8), the correct $k$ (for given $k_0$ and $p_F$) is the *smaller* of the two values deduced from (13) and (14), respectively. In (14), $\omega$ is of course directly related to $k_0$.

For any pair of values $k_0$ and $k$, straightforward algebra gives $p_F^2$. In Fig. 2 we have plotted $p_F^2$ versus $k^2$, for six different values of $\omega$, from 1.123 to 1.6, mostly in steps of 0.1. All quantities, $\omega$, $k$, and $p_F$, are in units of $\mu$. The dashed curves $A$ give the result of the simple Eq. (13); they are of course simple parabolas, vertically shifted relative to each other. The solid curves $B$ represent Eq. (14). As previously stated, for every $\omega$ and $p_F$ we should take the smallest $k$, i.e., the left-most curve. Taking, e.g., $\omega=1.4$, and starting from $p_F=0$, we start on the solid curve $B$ in which $k$ increases at first very slowly, then much faster. At $p_F^2=1.89$ and $k=1.60$, curves $A$ and $B$ intersect, and for higher $p_F$ we must then choose curve $A$. Similar behavior is shown for $\omega=1.5$ and 1.6 but curve $A$ becomes valid at much lower $p_F$ for these (circles in Fig. 2). At about $\omega\geq1.75$, curve $A$ becomes valid immediately at $p_F=0$.

For $\omega<1.4$, there is a somewhat peculiar behavior: Curve $B$ reaches its maximum before (i.e., at a lower $k$) it intersects curve $A$. The descending part of curve $B$ has, in our opinion, no physical meaning. We should follow curve $B$ to its maximum, then jump at constant $p_F$ to curve

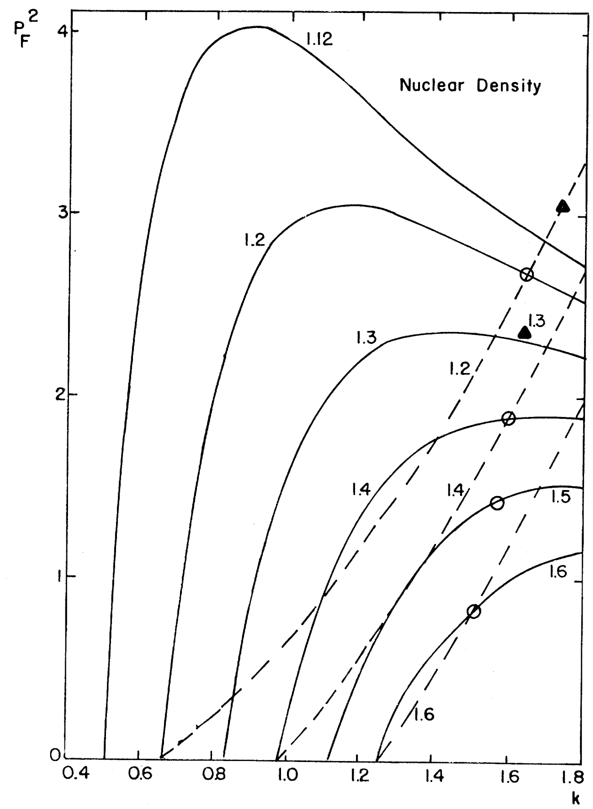


FIG. 2. The wave number $k$ of a pion (abscissa) as a function of $p_F^2$ (Fermi momentum squared) of nuclear matter (ordinate), for various pion energies $\omega$ (numbers attached to curves). All quantities in units of the pion mass $\mu$. The solid curves are from the Chew-Low theory (14), the dashed curves from the unitarity relation (13). Intersections of the two curves for a given $\omega$ are indicated by circles. For $\omega=1.2$ and 1.3, a triangle shows the lowest $p_F$ and $k$ for which the dashed curve is applicable. For the use of these curves, see the text below (14).

$A$ (indicated by dashes in Fig. 2). For $\omega=1.2$, this means that at $p_F^2=3.06$, $k$ jumps from 1.15 to 1.74. While this behavior is odd, it is no worse than a potential jump in ordinary Schrödinger theory, which is easily tractable. When the gradual change of density at the surface of a nucleus is taken into account, our jump may be smoothed out. It occurs only at $\omega<1.4$, i.e., pion kinetic energies < 56 MeV which are difficult to handle experimentally. For $\omega<1.15$ (kinetic energy 21 MeV), the jump occurs at a density higher than nuclear matter density and is therefore not important.

Curve $A$ corresponds to the $\pi$ being (essentially) at resonance, since $\delta>45°$. While resonance "begins" at $\omega\approx1.75$ for free nucleons, its beginning is shifted to lower and lower $\omega$ as the density increases. This is the equivalent of pressure broadening in optics. At nuclear matter density, the resonance region begins at $\omega = 1.15$! In the literature, a downward shift of the resonance has been reported; we believe the most important effect is broadening.

Once we are on curve $A$, the Pauli factor (13) will soon become 1. We have used (12a); we must stop doing so when $k = 4p_F/3$; from then on $k$ is given by (11) and is slightly smaller (in this region) than (13).

The *imaginary* part of $f(0)$ is proportional to the actual scattering cross section, i.e., to the probability of scattering with a *change* of direction, $\vec{k} \to \vec{k}'$. In nuclear matter, because of conservation of momentum, this means that a nucleon must change its state, i.e., we have *inelastic*[11] scattering. The probability for this is decreased by the Pauli factor (12) in which $k$ is replaced by the momentum change

$$q = |\vec{k}' - \vec{k}| = 2k \sin(\theta/2) \tag{15}$$

so that small-angle inelastic scattering is suppressed. The energy of the scattering nucleon must, of course, increase *more* than for a free nucleon, because in nuclear matter the potential energy of a nucleon increases with increasing momentum: This further decreases the probability of inelastic pion scattering, but probably this decrease is significant only for low pion energy.

In any case, $\mathrm{Im}f(0)$ must be calculated from the inelastic scattering, and will generally be much smaller than $i \sin^2(\delta/k)$ [cf. (6)] if $\delta$ is calculated from the Chew-Low formula (7).

An interesting feature of the unitarity requirement is that the inelastic scattering is, in any case, limited (for the 33 state) by

$$\sigma_{\mathrm{inel}}(n\pi^-) < 2\pi/k^2. \tag{16}$$

Since $k$ is, in most practical cases, of the order of $p_F$ or larger, and using $p_F = 1.34\ \mathrm{F}^{-1}$ for nuclear matter, this means

$$\begin{aligned} &\sigma_{\mathrm{inel}}(n\pi^-) < 30\ \mathrm{mb}, \\ &\sigma_{\mathrm{inel}}(p\pi^-) < 10\ \mathrm{mb}. \end{aligned} \tag{17}$$

Thus the cross section per nucleon is kept somewhat below its "geometrical cross section," $\pi r_0^2$ in the nucleus, even at resonance.

The main theoretical work now has to begin, in particular comparison with data. Our theory is merely qualitative, but its qualitative features differ from any existing theory we know. Our main results are the following: (1) The wave number $k$ of a pion in a nucleus never becomes unduly large, regardless of the nuclear density. (2) The resonance is very much broadened by density. (3) Inelastic scattering is appreciably suppressed.

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