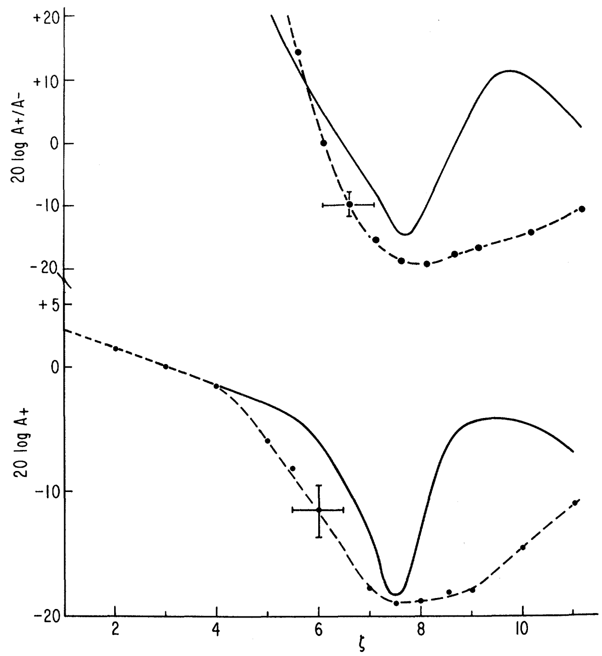

FIG. 4. Bottom curves, ratio of the test wave's amplitude to the main wave's amplitude. Top curves, ratio of the test wave's amplitude to the satellite wave's amplitude. Theoretical curves (solid) obtained using Eq. (5); data (dots and dashed curves) obtained from Fig. 3.

the waves at adjacent unstable frequencies no longer grow according to linear theory. We have observed small test waves at neighboring unstable frequencies and predict their behavior by regarding them as a slow modulation of the main wave's amplitude and phase.

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Model of Parametric Excitation by an Imperfect Pump*

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We examine the three-wave decay instability due to a monochromatic pump with a phase which undergoes a random walk with diffusion coefficient D. Analytic results are obtained for small and large D/γ_0 (where γ_0 is the growth rate for $D=0$), which demonstrate, respectively, a fractional reduction of γ by D/γ_0 and a growth rate γ_0^2/D. Results are also presented for intermediate values of D/γ_0.

It is now clear that parametric instabilities play a crucial role in the interaction of intense radiation with plasma.[1] Some of the effects are desirable, such as the anomalous absorption and concomitant heating in confined plasmas. However, in the laser-pellet interaction, this heating, if untimely, may prove troublesome for fusion prospects.[2] In addition the presence of the Raman and Brillouin backscattering instabilities[3] in the highly underdense region may, theoretically at least, isolate the core of the pellet from the radiation.

Because of the resonant nature of some of these instabilities, any mechanism which causes the pump to become nonmonochromatic is expected to reduce or even throttle the growth of these instabilities. One approach which might be used for the control of these instabilities is by making laser-produced radiation noisy. We consider here a model problem which treats one aspect of the noisy nature of this radiation. In this model we consider the electromagnetic pump of the form $E_0\cos(\omega_0 t-\varphi)$ and treat its stochastic properties as arising from a series of small random deflections in its phase φ, but regard the amplitude and frequency fixed. That is, we take

$$\dot{\varphi}(t)=\sum\delta\varphi_n\,\delta(t-t_n). \tag{1}$$

To our knowledge, the only other work along these lines has been that of Bodner,[4] who considered a frequency-modulated pump, and that of Thomson[5] who treated a pump whose amplitude autocorrelation is Gaussian.

In this work we discuss the three-wave decay instability of the pump into two normal modes of the plasma. (We treat here, for concreteness, the decay of the electromagnetic pump into an electron-plasma oscillation and an ion-acoustic wave, but with trivial modification the general three-wave problem can be subject to the same analysis.) We begin by writing down the pair of coupled oscillator equations[6] for n_1 and n_2, the density fluctuations in $\vec{k}$ space associated with ion-acoustic wave and electron-plasma oscillation, respectively:

$$\begin{aligned}\left(\frac{d^2}{dt^2}+2\nu_1\frac{d}{dt}+\omega_1^2\right)n_1&=-iCn_2\cos[\omega_0 t-\varphi(t)],\\ \left(\frac{d^2}{dt^2}+2\nu_2\frac{d}{dt}+\omega_2^2\right)n_2&=iCn_1\cos[\omega_0 t-\varphi(t)].\end{aligned} \tag{2}$$

Here ν_1 and ν_2 are linear damping decrements, and the coupling coefficient $C=\vec{k}\cdot\vec{E}(e^2/m_1m_2)^{1/2}$.

To order ν^2/ω^2 these equations may be factorized into

$$\begin{aligned}(d/dt+i\omega_1+\nu_1)(d/dt-i\omega_1+\nu_1)n_1&=-iCn_2\cos(\omega_0 t-\varphi),\\ (d/dt+i\omega_2+\nu_2)(d/dt-i\omega_2+\nu_2)n_2&=iCn_1\cos(\omega_0 t-\varphi).\end{aligned} \tag{3}$$

When we consider the interaction of just two of the modes near $-\omega_1$ and $+\omega_2$, this set reduces to the coupled first-order equations

$$\begin{aligned}(d/dt+i\omega_1+\nu_1)n_1&=(C/2\omega_1)n_2\cos(\omega_0 t-\varphi),\\ (d/dt-i\omega_2+\nu_2)n_2&=(C/2\omega_2)n_1\cos(\omega_0 t-\varphi).\end{aligned} \tag{4}$$

Here we take $\omega_1,\ \omega_2>0$, for a given $\vec{k}$. The additional following transformation,

$$\begin{aligned}n_1&=\tilde{n}_1(\omega_1)^{-1/2}\exp\{[i(\omega_2-\omega_1)-(\nu_1+\nu_2)]\tfrac{1}{2}t\},\\ n_2&=\tilde{n}_2(\omega_2)^{-1/2}\exp\{[i(\omega_2-\omega_1)-(\nu_1+\nu_2)]\tfrac{1}{2}t\},\end{aligned} \tag{5}$$

symmetrizes the equations into the following forms

$$\begin{aligned}[d/dt+\tfrac{1}{2}\delta\nu+\tfrac{1}{2}i(\omega_0-\Delta)]\tilde{n}_1&=F\tilde{n}_2\cos(\omega_0 t-\varphi),\\ [d/dt-\tfrac{1}{2}\delta\nu-\tfrac{1}{2}i(\omega_0-\Delta)]\tilde{n}_2&=F\tilde{n}_1\cos(\omega_0 t-\varphi).\end{aligned} \tag{6}$$

Here $\delta\nu\equiv\nu_1-\nu_2$, $\Delta\equiv\omega_0-\omega_1-\omega_2$, and $F=C/2(\omega_1\times\omega_2)^{1/2}$. The equations for the sum and difference of $\tilde{n}_1$ and $\tilde{n}_2$ satisfy

$$\begin{aligned}dS/dt+i\Omega D&=FS\cos(\omega_0 t-\varphi),\\ dD/dt+i\Omega S&=-FD\cos(\omega_0 t-\varphi),\end{aligned} \tag{7}$$

where now $\Omega=(\omega_0-\Delta-i\,\delta\nu)/2$. If we now eliminate D and discard nonresonant terms and terms of $O(F)$, we obtain

$$d^2S/dt^2+[\Omega^2+F\omega_0\sin(\omega_0 t-\varphi)]S=0. \tag{8}$$

In general Ω is complex. We temporarily restrict our analysis to the case where the imaginary part of Ω is unimportant, i.e., to the case where the growth rates are greater than the difference of the linear damping decrements. In this case we have the standard form of the Mathieu equation with a small pump of frequency close to twice that of the driven oscillator. If we define dimensionless long-time scale variables

$$t'=Ft,\quad \Delta'=\Delta/F, \tag{9}$$

and assume a solution of the form

$$S=R(t')\sin[\tfrac{1}{2}\omega_0 t-\chi(t')], \tag{10}$$

we obtain the following set of coupled equations on the long-time scale:

$$\dot{R}/R=\tfrac{1}{2}\cos\beta,\quad \dot{\beta}+\sin\beta=\dot{\varphi}-\Delta, \tag{11}$$

where $\beta=\varphi-2\chi+\pi$. In the absence of any fluctuation in the phase, the solution relaxes to $\beta=\beta_0\equiv\arcsin(-\Delta)$, $\dot{R}/R=\gamma_0\equiv\tfrac{1}{2}(1-\Delta^2)^{1/2}$, as t approaches infinity. The effect of the phase fluctuations is to prevent this relaxation from becoming complete, i.e., $\dot{\varphi}$ is a forcing term which scatters β away from its asymptotic value. It is a competition between this scattering resulting from the stochastic nature of $\dot{\varphi}$ and the relaxation of β to β_0 which determines the ensemble average growth rate

$$\langle\gamma\rangle\equiv\tfrac{1}{2}\int_0^{2\pi}d\beta\,W(\beta)\cos\beta. \tag{12}$$

Here $W(\beta)$ is the probability density for the distribution of β which is to be found. The nonlinear stochastic equation for the random variable β is then

$$\dot{\beta}+\sin(\beta+\beta_0)-\sin\beta_0=\dot{\varphi}, \tag{13}$$

where we now measure β from its steady value β_0 in the absence of noise, $\sin\beta_0=-\Delta$. If we now formally integrate for a time Δt long enough so that φ has made many small jumps but short enough that β has not changed appreciably, then

$$\Delta\beta+[\sin(\beta+\beta_0)-\sin\beta_0]\Delta t=\Delta\varphi+O(\Delta t^2). \tag{14}$$

Therefore the transition probability of the left-hand side is the same as that for $\Delta\varphi$, which is known from the theory of random flights,[7]

$$\psi(\Delta\beta;\beta,\Delta t)=(4\pi D\Delta t)^{-1/2}\exp(-\{\Delta\beta+\Delta t[\sin(\beta+\beta_0)-\sin\beta_0]\}^2/4D\Delta T), \tag{15}$$

where $D=n\langle\delta\varphi^2\rangle/2$, n is the number of displacements in phase per unit time, and $\langle\delta\varphi^2\rangle$ is the mean square displacement in phase expected on any given occasion. It is readily shown that this transition probability implies the power spectrum of the incident radiation to be $S(\omega)=E_0{}^2D/2[D^2+(\omega-\omega_0)^2]$.

With the knowledge of the transition probability ψ we can obtain the Fokker-Planck equation[7] for the distribution W of β,

$$\partial W(\beta,t;\beta_0,D)/\partial t=-\partial(\langle\Delta\beta\rangle W)/\partial\beta+\tfrac{1}{2}\partial^2(\langle\Delta\beta^2\rangle W)/\partial\beta^2. \tag{16}$$

Here

$$\langle\Delta\beta\rangle=(\Delta t)^{-1}\int\Delta\beta\,\psi(\Delta\beta,\beta)\,d\Delta\beta=-[\sin(\beta+\beta_0)-\sin\beta_0], \tag{17a}$$

$$\langle\Delta\beta^2\rangle=(\Delta t)^{-1}\int\Delta\beta^2\psi(\Delta\beta,\beta)\,d\Delta\beta=2D. \tag{17b}$$

It is the stationary, periodic solution of this equation we are concerned with, that is, the solution of

$$D\,\partial^2W/\partial\beta^2+(\partial/\partial\beta)\{[\sin(\beta+\beta_0)-\sin\beta_0]W\}=0, \tag{18}$$

with $W(\beta+2\pi)=W(\beta)$. This equation is readily solved to yield

$$W=C\exp[a\cos(\beta+\beta_0)]\int_0^\infty\exp(-ta\sin\beta_0)\exp[-a\cos(t+\beta+\beta_0)]\,dt, \tag{19}$$

with the constant C to be determined from the normalization $\int W\,d\beta=1$, and $a\equiv 1/D$. We restrict β_0 to the values $0\leqslant\beta_0<\pi/2$.

We are now prepared to write an expression for the modified growth rate:

$$\langle\gamma\rangle=\tfrac{1}{2}\int W\cos(\beta+\beta_0)\,d\beta/\int W\,d\beta, \tag{20}$$

or

$$\langle\gamma\rangle=\frac{1}{2}\frac{\int_0^\pi\exp(-2at\sin\beta_0)I_1(2a\sin t)\sin t\,dt}{\int_0^\pi\exp(-2at\sin\beta_0)I_0(2a\sin t)\,dt}, \tag{21}$$

where the I's are Bessel functions of the second kind. The asymptotic values are readily obtained for large and small D. For $D\gg1$,

$$\langle\gamma\rangle\cong(4D)^{-1}, \tag{22}$$

while for $D\ll1$,

$$\langle\gamma\rangle\cong\tfrac{1}{2}\cos(\beta_0)(1-D/2\cos^3\beta_0). \tag{23}$$

These asymptotic results are plotted together with the exact value of $\langle\gamma\rangle$ versus D for $\beta_0=0$ in Fig. 1. The growth rate decreases monotonically with increasing D. Note that the asymptotic results are a good approximation to $\langle\gamma\rangle$ even for finite D.

We have thus found the average growth for the three-wave decay process in a plasma when the pump is subject to random alterations of phase. Such a nonideal effect can cause significant reduction of the growth rate when the phase diffusion time is comparable to or shorter than the growth time. We realize that to make noisy lasers goes counter to their more usual attractive feature, i.e., their coherence, but it may be necessary to trade off this feature in order to throttle the undesirable instabilities.

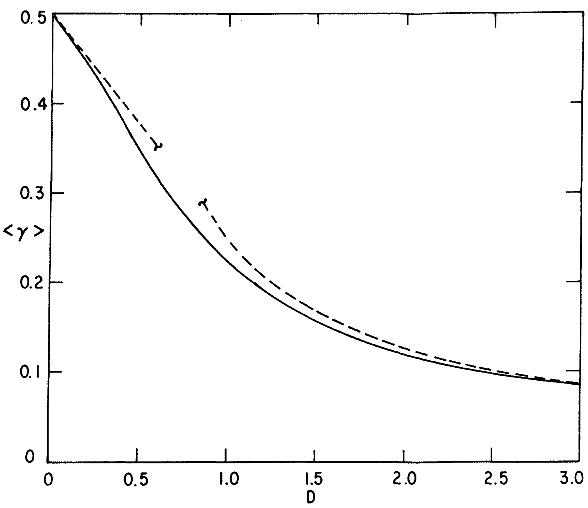

FIG. 1. Ensemble average growth rate $\langle\gamma\rangle$ versus the phase diffusion coefficient D. Solid line, exact result, calculated from Eq. (21); dashed lines, asymptotic results for large and small D, as given in Eqs. (22) and (23).

We remark here that the derivation of Eq. (8) is unchanged if we regard F and ω_0 as slowly varying but otherwise arbitrary functions of time. Thus the analysis of the effects of amplitude and frequency uncertainty (e.g., as occurs for a multimode pump) can be picked up at this point. This will be discussed in a future paper.

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Tricritical Points of Thin Superconducting Films*

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A similarity between the phase diagrams of systems which are believed to exhibit tricritical points and the phase diagram of the transition to the superconducting state of a thin film in a uniform magnetic field suggests the existence of a tricritical point in the latter system. The superconducting case is of interest because the field thermodynamically conjugate to the order parameter can be realized physically. The conjugate field plays an important role in models of the tricritical point but has been believed to be unphysical in systems studied thus far.

The purpose of this Letter is to point out a similarity between the phase diagram of a thin superconducting film in a magnetic field and the phase diagrams of systems exhibiting tricritical points.[1-5] This similarity suggests the existence of a tricritical point in the superconducting case. This would be significant because in superconductors it is possible by means of electron tunneling to realize the field thermodynamically conjugate to the order parameter associated with the critical line of the phase diagram.[6-8] This field is important in current models of the tricritical point[1] in other systems, but is unphysical. In addition, the classical second-order nature of the phase transition[9,10] in superconductors should permit realistic calculations of properties in the vicinity of the tricritical point.

The essential experimental feature of a system possessing a tricritical point is that there is a point at which a higher order of λ transition line in the space of thermodynamic fields (temperature, pressure, etc.) becomes a first-order transition line.[1] In Fig. 1(a) we show the phase diagram for a metamagnet. The figure caption con-