Intermittency in Fully Developed Turbulence as a Consequence of the Navier-Stokes Equations*

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Fully developed hydrodynamic turbulence is considered in terms of a steady-state solution to the exact Fokker-Planck equation for the probability functional of the velocity field. The nested intermittency argument of Yaglom is restated in this formulation. A divergent buildup of fluctuations during the energy cascade modifies the $k^{-5/3}$ inertial subrange spectrum. A physical analogy of the critical exponent η to the Yaglom intermittency parameter $(\mu/9)$ is proposed.

Fully developed hydrodynamic turbulence is characterized in three dimensions by a cascade of energy from small to large wave numbers. The nonlinear inertial terms drive this cascade which eventually terminates at large wave number as a result of viscous dissipation. The cascade is characterized by three parameters: ϵ, the net rate at which energy is dissipated per unit mass; l, the characteristic length scale at which the turbulent energy is put into the fluid; and ν, the kinematic viscosity of the fluid. From ϵ and ν one can form the characteristic length $\eta = \nu^{3/4}\epsilon^{-1/4}$ which determines the scale on which dissipation takes place. Very large values of l/η correspond to very large Reynolds number and are characteristic of most oceanic and atmospheric flows. In such flows there is a wide range of wave numbers $l^{-1} \ll k \ll \eta^{-1}$ in which both energy input and energy dissipation are negligible. In this "inertial subrange" one might expect a universal form for the energy spectrum $E(k)$. If this function were to be independent of both l and ν, and to depend only on ϵ, then by dimensional arguments it would be given by $E(k) = c\epsilon^{2/3}k^{-5/3}$. This was the form proposed by Kolmogorov and Obukhov[1] in 1941.

In 1962, Kolmogorov and Obukhov[2] proposed a modified form of their theory in which the stirring length played an essential role, but the viscosity was still unimportant. The essential feature of the modified theory is that the average dissipation ϵ_r over a cube of side r should have a log-normal probability distribution with variance proportional to $\ln(l/r)$. The original Kolmogorov theory, with $\epsilon^{2/3}$ replaced by $\langle \epsilon_r^{2/3} \rangle$, then gives a modified energy spectrum. In 1966 Yaglom[3] gave an argument for the intermittency of ϵ_r characterized by the log-normal probability distribution. There is now a considerable amount of experimental data supporting the Yaglom[4] model, but there are difficulties in its theoretical justification. Though the entire argument is correctly based on the absence of an internal length scale, there is no reference to the underlying Navier-Stokes equations. For a wide range of applications to intermittent phenomena this is an advantage,[5] but for a physical understanding of hydrodynamic turbulence it is a severe limitation. Furthermore there is no theoretical reason why the probability distribution of ϵ_r should be simply related to the energy spectrum $E(k)$ even though the former is log-normal.

We work with the Fourier transform of the Navier-Stokes equations for isotropic homogenous turbulence in an incompressible fluid. The system is driven by a statistically defined external force. Since the detailed properties of the external force should not be relevant, let us make the simplification that it is a Gaussian process with correlation function

$$\langle F_k^{\alpha}(t) F_k^{\beta}(t') \rangle = 2h_k \delta(t-t') \delta^{\alpha\beta}. \quad (1)$$

The equation of motion for the kth Fourier component $u_k^{\alpha}(t)$ of the transverse velocity fluctuation is then a nonlinear Langevin equation of the same type studied by Zwanzig[6] or Kawasaki[7] in connection with time correlation functions at equilibrium. This equation can be transformed by standard methods to a Fokker-Planck equation for the probability functional $P(\{u_k\}, t)$ of the velocity field. This equation, previously derived by Edwards and McComb,[8] is

$$\frac{\partial P}{\partial t} + \sum_k \frac{\partial}{\partial u_k^{\alpha}} \Big(\nu k^2 u_k^{\alpha} - \sum_{j,l;\beta,\gamma} M_{kjl}^{\alpha\beta\gamma} u_j^{\beta} u_l^{\gamma} + h_k \frac{\partial}{\partial u_{-k}^{\alpha}} \Big) P = 0, \quad (2)$$

where

$$D_k^{\alpha\beta}=\delta^{\alpha\beta}-k^\alpha k^\beta|k|^{-2}, \quad (3)$$

$$M_{kjl}^{\alpha\beta\gamma}=\frac{\Delta}{i(2\pi)^3}(k^\beta D_k^{\alpha\gamma}+k^\gamma D_k^{\alpha\beta})\delta_{kjl}, \quad (4)$$

$\Delta=(2\pi/L)^3$; $\delta_{kjl}=1$ if $\vec{k}+\vec{j}+\vec{l}=0$, zero otherwise. In the limit of $L\to\infty$, Δ becomes d^3k and

$$\delta_{kjl}\to\Delta\delta(\vec{k}+\vec{j}+\vec{l}).$$

A time-independent solution is looked for. Compare this solution with the thermal-equilibrium case considered in Refs. 6 and 7. In that case there is a fluctuation-dissipation theorem, $h_k=\nu k^2$. This makes the steady-state solution a product of Gaussians $\exp(-\frac{1}{2}\sum_{k\alpha}u_k^\alpha u_{-k}^\alpha)$ even in the presence of the nonlinear terms. When the h_k drive the system far from equilibrium, however, the steady-state probability is Gaussian only in the limit of small Reynolds number where the nonlinear terms can be neglected. For large Reynolds number far from equilibrium the probability distribution of the u_k is far from Gaussian. The main point of this paper is that this distribution becomes increasingly intermittent with increasing k, with a variance which grows proportional to $\ln(kl)$.

The essential simplifying assumption here is to let the viscosity approach zero, and to assume that the probability functional is unaffected except in the limit of infinite k where the dissipation will occur. For convenience the fluid is driven at a single wave number so that

$$h_k=(\epsilon/4\pi k^2)\,\delta(k-l^{-1}).$$

Let us introduce the dimensionless variables

$$q=kl,\quad v_q=u_k\epsilon^{-1/3}l^{-10/3}. \quad (5)$$

(The explicit indication of Cartesian components will be omitted from this point on.) This eliminates the dependence on ϵ, but the scaling behavior with q must still be examined. The n-mode probability distribution is

$$p(v_1,v_2,\ldots,v_n)=\delta(\vec{q}_1+\vec{q}_2+,\ldots,+\vec{q}_n)\times p'(v_1,v_2,\ldots,v_{n-1}), \quad (6)$$

where the first factor is due to translational invariance. In the original Kolmogorov theory, the distribution p' is invariant to the scale transformation

$$q\to sq,\quad v_q\to s^{-10/3}v_q. \quad (7)$$

This leads to the energy spectrum $E(q)=Cq^{-5/3}$. The exponent $-\frac{5}{3}$ has the following contributions: $+2$ from the definition of $E(q)$ as energy per unit wave number, $+3$ from the δ function in Eq. (6), -6 from the two volume integrals in the Fourier transforms, and $-\frac{2}{3}$ from the scaling property of the velocity $u(r)$.

By examining the cascade through a large range of wave numbers one can plausibly argue that $p'(v_q)$ does not have the scaling behavior of Eq. (7). In fact the distribution becomes increasingly intermittent with increasing q in such a way as to modify the exponent in the expression $E(q)=Cq^\alpha$ but still maintain the power-law form. The argumentis essentially the same as given by Yaglom for $p(\epsilon_r)$.

Let $q=c^N$, where $c>1$ and N is a large number. Consider an intrinsically positive random variable such a $y_N=v_qv_{-q}$. Introduce the ratios $e_j=y_{N-j+1}/y_{N-j}$, where y_n is associated with wave number $q'=c^n$, and write

$$y_N=y_q=e_1e_2e_3\cdots e_Ny_1$$

so that

$$\ln y_q=\sum_{j=1}^{N}\ln e_j+\ln y_1.$$

Because there is no internal length scale, the probability distribution of each of the e_j should be identical for all except a few e_j, with j of the order of N. If, in addition, the probability distributions of the e_j are nearly independent, the central-limit theorem applies so that $\ln y_q$ is normally distributed. The mean of $\ln y_q$ is Nm_1 and its variance is $N\sigma_1^2$, where m_1 and σ_1^2 are the mean and variance of each of the e_j. Recalling that $N=(\ln q/\ln c)$, and arguing that the final result should not depend on the scale factor c, the mean and variance of $\ln y_q$ are given by $m=a\ln q$ and $\sigma^2=b\ln q$. The constants a and b are universal, and are to be determined by a more complete theory.

To calculate $E(q)$ one notes that the moments of a log-normal distribution are given by[2]

$$\langle y_q^n\rangle=\exp(nm+\tfrac{1}{2}n^2\sigma^2)$$

so that

$$\langle y_q\rangle=q^\beta,\quad \beta=(a+\tfrac{1}{2}b).$$

To estimate β let us go back to coordinate space and assume that the quantity

$$\Delta u^2(r)=[u(\vec{x}+\vec{r})-u(\vec{x})]^2$$

is also log-normally distributed with mean and variance proportional to $\ln(l/r)$. Note then that the third moment $\langle\Delta u^3(r)\rangle$ must be given correctly by the original Kolmogorov theory in order that the total energy cascading through the inertial subrange be constant. (See Ref. 1, Sec. 33.) This implies a relation between the mean and variance in r space such that[3,4]

$$\langle \Delta u^n(r)\rangle = C_n \epsilon^{n/3} r^{n/3-\mu n(n-3)/18}$$

and the energy spectrum is

$$E(q) = Cq^{-(5/3+\mu/9)},$$

where μ is the Yaglom parameter. Experimentally,[4] $\mu = \frac{1}{2}$ to within about 10%.

Clearly, the argument leading to the modified power law for $E(q)$ is far from rigorous. It has already been pointed out that there are problems of determinacy[9] and internal inconsistency[5] in the Yaglom log-normal model, and the present argument does nothing to correct these deficiencies. More recent work shows how modified power laws can arise without invoking an approximate log-normal model.[10] The main physical point is that a divergent buildup of fluctuations during the cascade can modify the Kolmogorov $k^{-5/3}$ spectrum even when the energy cascade is local in k space. The energy cascade is a multiplicative random process capable of long-range effects. This type of long-range effect due to a short-range force is familiar in critical phenomena, and is exhibited in the critical exponent η. It is tempting to identify this with the quantity $\mu/9$ in the present context. The fact that $\mu/9$ is approximately 0.05 for three-dimensional turbulence is very suggestive. The kind of scaling argument used in our crude model suggests that a more rigorous argument might be possible using renormalization-group techniques, but we have not seen how to carry this out. Since the parameter μ has been measured experimentally, and its existence seems to be a plausible consequence of the Navier-Stokes equations, such a calculation would be of considerable interest.

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