Stability of Plane Shock Waves*

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A new argument is presented that yields the following criterion for plane shocks to be stable: $-1<j^2(dv/dp)_{\rm H}<1$, where $j^2=(p-p_0)/(v_0-v)$ is the slope of the Rayleigh line, and $(dv/dp)_{\rm H}$ is the slope of the Hugoniot curve in the pressure-volume plane. The lower limit is well known and the consequences of its violation are well understood; however, no such degree of understanding has yet been achieved for the upper limit. It seems likely that it bears an important relation to detonation phenomena.

We have recently re-examined some of the theoretical results on the stability of plane shocks: This work included a careful check by one of us (G.W.S.) of an analysis by D'yakov.[1] In attempting to understand the conclusions we conceived a new approach to the problem that leads to a different stability limit than has previously been derived.

In the analysis of D'yakov, and also that of Erpenbeck[2] whose mathematical technique was different, a plane steady shock is perturbed and the growth with time of the perturbation quantities are examined via the one-dimensional flow equations in linearized form. The stability limits can be summarized as

$$-1<j^2(dv/dp)_{\rm H}<1\pm 2|M|, \tag{1}$$

where $j^2=(p-p_0)/(v_0-v)$ is the slope of the Rayleigh line, $(dp/dv)_{\rm H}$ is the slope of the Hugoniot curve in the pressure-volume (p-v) plane, and M is the Mach number of the shock with respect to the material behind,

$$M=(D-u)/c,$$

with D the shock speed, u the particle velocity, and c the local sound speed.

The ambiguity in sign of the upper limit of Eq. (1) has not been resolved. D'yakov's analysis gives both signs, although he evidently rejected the positive sign. Erpenbeck's result is stated differently, but when cast in the above form his treatment appears to yield only the positive sign. This latter solution has been shown by Gardner to correspond to the limit for breakup of a single shock into two waves propagating in opposite directions, and is therefore probably the correct bound for the problem as posed.[3] The lower limit is well known to correspond to breakup into two waves propagating in the same direction.[4,5]

If perturbations of the boundary conditions are considered as well as perturbations of the shock, however, we are led to a different stability limit. To see this we map the Hugoniot curve in the pressure-particle-velocity (p-u) plane by means of the formula derived from the jump conditions,

$$(dp/du)_{\rm H}=2j[1-j^2(dv/dp)_{\rm H}]^{-1}. \tag{2}$$

Points ABC in the p-v plane of Fig. 1 then map onto points ABC of Fig. 2 in the p-u plane. Of

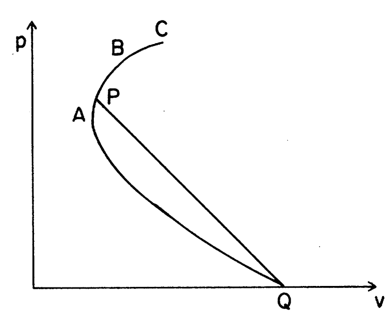

FIG. 1. Hugoniot curve in the p-v plane; Rayleigh line PQ.

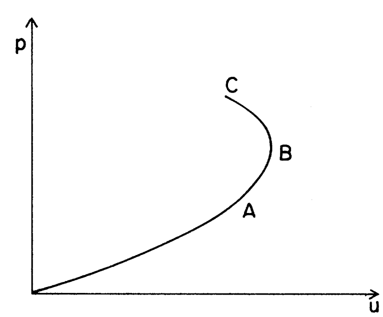

FIG. 2. Hugoniot curve of Fig. 1 in the p-u plane.

special interest is the region for which $(du/dp)_{\rm H} < 0$ for in this region two solutions are possible for the common p-u state produced by impact of a projectile with a target.

Consider a plane uniform shock with pressure p_1 perturbed at the piston boundary $x = 0$ at time t_1 by an incremental pressure $\delta p = p_2 - p_1$, as indicated in Fig. 3. The increased pressure is held indefinitely on the boundary after t_1 and is transmitted into the shocked region by a forward-facing sound wave C^+ propagating with characteristic velocity $dx/dt = u + c$. Upon reflection at the shock front a backward-facing wave C^- propagates toward the piston boundary with speed $u - c$. Along each of these characteristic paths a compatibility condition obtains:

$$\Gamma^+\!: \; dp = -\rho c\, du \text{ on } C^+;$$

$$\Gamma^-\!: \; dp = \rho c\, du \text{ on } C^-.$$

Transitions between each of the numbered states of Fig. 3 can occur only along lines of one or the other of these families of curves, Γ^+ or Γ^-. Moreover, continuity of pressure and particle velocity requires that each state lie on either the prescribed p-u locus at the shock front (Hugoniot), or the line $p_2 = \text{const}$ $(t > t_1)$.

Figure 4(a) shows the predicted behavior in the pressure-particle velocity plane when the slope of the Hugoniot $(dp/du)_{\rm H} > 0$. In this case the perturbation steadily diminishes, forming a kind of convergent spiral to a focus on the Hugoniot at pressure p_2.

Figure 4(b) shows a case when $dp/du < 0$. The spiral now diverges and the perturbation steadily grows with time. The spiral configuration obtains when the Hugoniot is steeper than the characteristics, as in Fig. 4. When the Hugoniot slopes are shallower the configurations consist of oscillating pressures but monotonically increasing or decreasing particle velocities. The same stability condition is valid, however.

Based on this argument the stability limits should be stated as

$$-1 < j^2(dv/dp)_{\rm H} < 1, \tag{3}$$

or, as is easily shown, an equivalent statement is

$$\left(\frac{p - p_0}{T}\right)\left(\frac{\partial T}{\partial p}\right)_S - 1 < j^2\left(-\frac{\partial v}{\partial p}\right)_S < 1, \tag{4}$$

where the subscript S indicates isentropic derivatives and T is the temperature.

This analysis has been simplified by neglecting reflection of the acoustic waves at the contact discontinuities that are produced each time

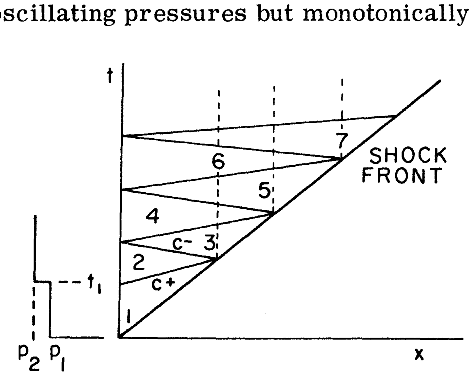

FIG. 3. Shock front and characteristics in x-t plane; vertical dashed lines are contact discontinuities.

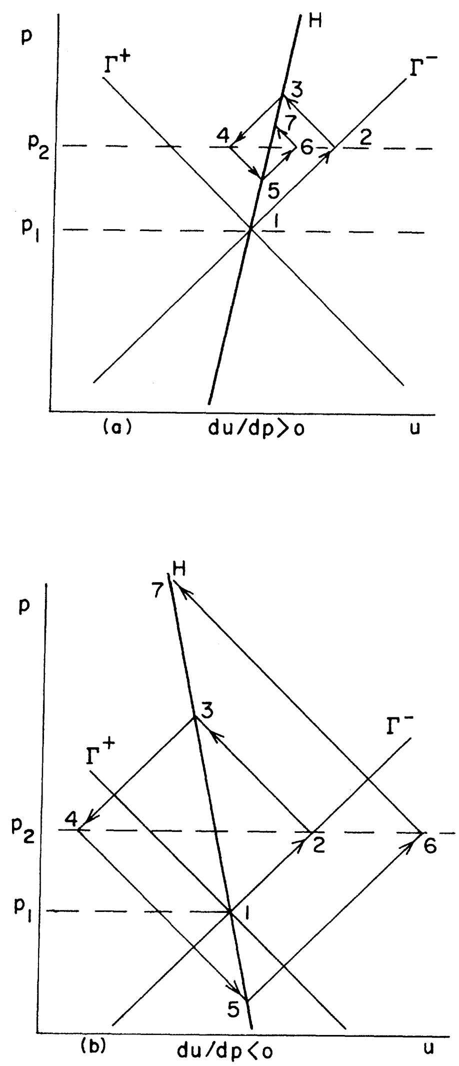

FIG. 4. Portion of Hugoniot curve H, and Γ characteristics in the p-u plane. Numbers correspond to those of Fig. 3. (a) Stable, (b) unstable.

there is a change in the amplitude of the shock. While these reflections alter the details of the process, we do not expect them to alter the conclusion; limited numerical experiments bear out this premise. Moreover, violation of the upper limit of Eq. (3) also implies that $(dE/dp)_H < 0$, where E is the internal energy. It is difficult to understand how a shock subject to this condition can attenuate since the internal energy would necessarily go through a maximum as the pressure decreased.

A more general treatment is under investigation; however, we tentatively conclude that a necessary condition for a shock to be stable is that its Hugoniot curve have positive slope in the p-u plane, or, equivalently, that the magnitude of its slope in the p-v plane be greater than that of the Rayleigh line or its mirror reflection about the vertical. Although the consequences of violation of the lower limit are well understood, no such degree of understanding has yet been achieved for the upper limit. It seems likely that it bears an important relation to detonation phenomena.

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Longitudinal Force Exerted by Circularly Polarized High-Powered Laser Radiation in a Dense Electron Plasma*

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A plane, circularly polarized electromagnetic wave of *finite* transverse extent must have a longitudinal component of electric field intensity, as a consequence of $\nabla\cdot\vec{E}=0$. The magnitude of the resulting oscillatory, longitudinal force on plasma electrons may be large compared with other longitudinal forces. The associated periodic bunching of electrons along the laser axis results in large collective plasma effects.

The purpose of this note is to illustrate the importance of longitudinal electric fields in circularly polarized laser beams of finite transverse extent. We calculate the steady-state solution for this case that can be compared with the infinite-plane-wave solution given by Steiger and Woods, hereafter referred to by SW.[1] Taking this steady state as the zeroth-order solution, we then show that important plasma effects arise, even for normal incidence. Linear polarization with normal incidence gives the electrons a drift velocity caused by $\vec{v}\times\vec{B}_{\text{laser}}$ which eludes steady-state solution; this term is zero for circular polarization. Other types of longitudinal forces have been studied previously for linearly polarized beams.[2,3]

A plane, monochromatic, circularly polarized electromagnetic wave propagating in the z direction may be represented by the field vectors

$$\vec{E}(z,t)=E(\hat{\epsilon}_1+i\lambda\hat{\epsilon}_2)e^{-i(\omega t-kz)},\qquad \vec{B}(z,t)=B(-i\lambda)(\hat{\epsilon}_1+i\lambda\hat{\epsilon}_2)e^{-i(\omega t-kz)}, \tag{1}$$

where the physical fields are taken as the real parts. The $\hat{\epsilon}_i$ are unit vectors in the x, y, and z directions, E and B are real quantities, and the helicity λ is $+1$ (-1) for left (right) circular polarization. As pointed out by Jackson,[4] limiting the transverse extent of the wave to be finite requires that the fields in Eq. (1) have a longitudinal component. In the absence of a net charge density, $\nabla\cdot\vec{E}=0$ yields

$$\vec{E}(\vec{r},t)=\left\{E(x,y)(\hat{\epsilon}_1+i\lambda\hat{\epsilon}_2)+\frac{i}{k}\left[\frac{\partial E}{\partial x}+i\lambda\frac{\partial E}{\partial y}\right]\hat{\epsilon}_3\right\}e^{-i(\omega t-kz)}. \tag{2}$$