[27]This beryllium shield was several times thicker, in radiation lengths, than similar shields used in other experiments.

[28]R. C. Miller, Phys. Rev. 95, 796 (1954).

A FORMAL OPTICAL MODEL

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A construction is given of an equivalent one-body potential for the elastic scattering of a particle incident on a complex target. We consider explicitly the case that incident and target particles are identical nonrelativistic fermions and allow fully for antisymmetry. The method is inspired by that of Frantz and Mills[1] and removes some defects therefrom; it also removes the basis for their proposed change in phenomenological optical model analysis. Center-of-mass motion is still ignored.

Denoting by $|\bar{\alpha}\rangle$ the scattering state and by $|\bar{0}\rangle$ the target ground state, we define the model wave function as

$$\phi(\vec{r}, t) = \langle \bar{0} | \bar{\psi}(\vec{r}, t) | \bar{\alpha}\rangle, \tag{1}$$

where $\bar{\psi}$ is the Heisenberg field operator of second quantization. For $|\bar{\alpha}\rangle$ we take the state

$$|\bar{\alpha}\rangle = \int_{-\infty}^{t} dt' e^{-iEt'} \bar{\psi}^\dagger(\vec{r}', t') |\bar{0}\rangle, \tag{2}$$

which corresponds to a source of particles of energy E at the point $\vec{r}'$; if r' is sufficiently large only a plane wave actually reaches the target. In writing, with $x \equiv (\vec{r}, t)$,

$$\phi(x) = \int_{-\infty}^{+\infty} dt' e^{-iEt'} G(x, x'), \tag{3}$$

$$G = \langle \bar{0} | T(\bar{\psi}(x), \bar{\psi}^\dagger(x')) | \bar{0}\rangle, \tag{4}$$

we make no error by taking the time-ordered rather than the retarded product, for the surplus contribution depends on the possibility of absorbing a particle at $\vec{r}'$ from the state $|\bar{0}\rangle$ and so vanishes for large r'.

We construct G by a perturbation theory where in zero order the real forces are replaced by a fictitious one-body potential, in general nonlocal,

$$\int d\vec{r} d\vec{r}' \bar{\psi}^\dagger(\vec{r}, 0) U(\vec{r}, \vec{r}') \bar{\psi}(\vec{r}', 0).$$

An S matrix is defined by

$$\frac{\partial}{\partial t} S(t, t') = -iH'(t) S(t, t'),$$

$$S(t', t') = 1, \quad H'(t) = e^{iH_0 t} H' e^{-iH_0 t},$$

where H_0 is the zero-order Hamiltonian and the total Hamiltonian is $H_0 + H'$. In terms of interaction representation operators ψ, the Heisenberg operator $\bar{\psi}$ can be written

$$\bar{\psi}(x) = S^{-1}(t, 0) \psi(x) S(t, 0),$$

and we have the usual expression

$$G = \frac{\langle 0 | S(\infty, t) \psi(x) S(t, t') \psi^\dagger(x') S(t', -\infty) | 0\rangle}{\langle 0 | S(\infty, -\infty) | 0\rangle},$$

for $t > t'$, and similarly for $t < t'$, where $|0\rangle$ is the zero-order target ground state—assumed nondegenerate. Expanding above and below in powers of H', we have

$$G = \sum_{n=0}^{\infty} \frac{1}{n!} \int dt_1 \cdots dt_n \langle 0 | T\{H'(t_1) \cdots H'(t_n) \psi(x) \psi^\dagger(x')\} | 0\rangle \Big/ \sum_{n=0}^{\infty} \frac{1}{n!} \int dt_1 \cdots dt_n \langle 0 | T\{H'(t_1) \cdots H'(t_n)\} | 0\rangle. \tag{5}$$

We make a diagrammatic analysis of this following Hubbard,[2] using Wick's theorem and regarding $|0\rangle$ as "vacuum." The denominator has the effect simply of cancelling all diagrams in the numerator not linked to the terminal operators $\psi(x)$ and $\psi^\dagger(x')$; G is therefore the sum of linked diagrams only from the numerator. We call "improper" a linked diagram which falls into two disconnected parts on the removal of some particle (as distinct from interaction) line. Reasoning familiar in field theory leads to the integral equation

$$G(x,x') = G_0(x,x') - iG_0(x,x'')W(x'',x''')G(x''',x'), \tag{6}$$

where repeated arguments are integrated over, G_0 is the zero-order value, and $-iW$ is the sum of all proper linked diagrams—omitting factors for the terminal lines. Then from (3) with $t=0$,

$$\phi(\vec{r}) = \phi_0(\vec{r}) - iG_0(E,\vec{r},\vec{r}')W(E,\vec{r}',\vec{r}'')\phi(\vec{r}''), \tag{7}$$

where

$$W(E,\vec{r},\vec{r}') = \int d(t-t')\, e^{iE(t-t')}W(x,x'), \tag{8}$$

and likewise for G_0.

Now if u_n are a complete set of wave functions for the potential U, with eigenvalues E_0,

$$-iG_0 = \sum^{\text{unocc.}} \frac{u_n(\vec{r})u_n^*(\vec{r}')}{E-E_n+i\epsilon} + \sum^{\text{occ.}} \frac{u_n(\vec{r})u_n^*(\vec{r}')}{E-E_n-i\epsilon}.$$

Since all the occupied states are of negative energy, for positive E the sign of the infinitesimal $i\epsilon$ is unimportant in the last term. Thus

$$\phi(\vec{r}) = \phi_0(\vec{r}) + \sum_n \frac{u_n(\vec{r})u_n^*(\vec{r}')}{E-E_n+i\epsilon} W(E,\vec{r}',\vec{r}'')\phi(\vec{r}''). \tag{9}$$

This is just the integral equation for scattering by an added potential W, and therefore

$$V(E,\vec{r},\vec{r}') = U(\vec{r},\vec{r}') + W(E,\vec{r},\vec{r}') \tag{10}$$

is the total optical potential. The scattering amplitude averaged over an interval of energy can be obtained from $V(E+i\epsilon,\vec{r},\vec{r}')$ with ϵ finite. This and other details will be discussed elsewhere.

In conclusion we compare this account with that of Frantz and Mills. They define an optical model wave function by projecting on to the zero-order rather than the real target ground state:

$$\phi'(\vec{r},t) = \langle 0|\bar{\psi}(\vec{r},0)|\bar{\alpha}\rangle. \tag{11}$$

In the diagrammatic analysis it is then natural to use time-ordered rather than Feynman diagrams. As a result, for example, the two diagrams of Fig. 1 are regarded as distinct and the second as "proper." The proper parts of diagrams are now connected by forward lines only (particles) and not backward lines (holes). In the equation replacing (9) the n summation then runs over unoccupied states only, and the optical potential is therefore of the form

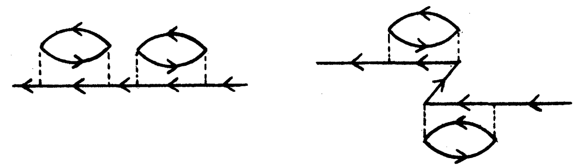

FIG. 1. Time-ordered diagrams.

$$U + PW',$$

where P is a projection operator on to unoccupied states. Even in the trivial case that the perturbation is only a change ΔU in the one-body potential, the "optical" potential based on U would have this form. In contrast, our development gives in this case the natural result $V = U + \Delta U$.

A more serious objection to the use of (11) as an optical model wave function is that it does not have an acceptable asymptotic form when inelastic scattering is possible. If $|\bar{n}\rangle$ denotes a state of excitation Δ_n of the target, then asymptotically

$$\langle \bar{n}|\bar{\psi}(\vec{r},0)|\bar{\alpha}\rangle \sim f_n r^{-1} e^{iK_n r},$$

with

$$K_n^2 = K_0^2 - 2M\Delta_n.$$

If

$$\langle 0| = \sum \xi_n \langle \bar{n}|,$$

then ϕ' has an outgoing part

$$\sum \xi_n f_n r^{-1} e^{iK_n r},$$

which is a superposition of inelastic as well as elastically scattered waves.

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[1]L. M. Frantz and R. L. Mills (to be published); Frantz, Mills, Newton, and Sessler, Phys. Rev. Letters 1, 340 (1958).

[2]J. Hubbard, Proc. Roy. Soc. (London) A240, 539 (1957).