NEW APPROACH IN THE THEORY OF SATELLITE ORBITS*

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If O is a center for the earth and if Oz points along the polar axis and Ox towards the vernal equinox, the oblate spheroidal coordinates ξ, η, ϕ, defined by

$$x = c[(\xi^2+1)(1-\eta^2)]^{1/2}\cos\phi, \quad (0<\xi<\infty)$$
$$y = c[(\xi^2+1)(1-\eta^2)]^{1/2}\sin\phi, \quad (-1\leq\eta\leq 1)$$
$$z = c\xi\eta,$$

turn out to be a useful system of coordinates for investigating the motion of an earth satellite.

In this coordinate system any axially symmetric potential $V(\xi,\eta)$ that results in separability of the Hamilton-Jacobi equation must have the general form

$$V(\xi,\eta) = (\xi^2+\eta^2)^{-1}[f(\xi)+g(\eta)].$$

The most general solution of Laplace's equation compatible with such a form is

$$V = C + b_0\,\mathrm{Re}(\xi+i\eta)^{-1} + b_1\,\mathrm{Im}(\xi+i\eta)^{-1} + b_2(\xi^2+\eta^2)^{-1}\left[2\xi\tan^{-1}\xi + \eta\ln\left(\frac{1+\eta}{1-\eta}\right)\right],$$

where the logarithmic term has a singularity everywhere on the z axis. We therefore place $b_2 = 0$ and we also place $C=0$ to make V vanish at infinity.

Expansion of $(\xi+i\eta)^{-1}$ in spherical harmonics leads naturally to the following choices for the adjustable constants b_0 and c:

$$b_0 c = -GM, \quad c^2 = (I_p - I_t)/M,$$

where G is the gravitational constant, M is the mass of the earth, and I_p and I_t are, respectively, the polar and transverse moments of inertia of the earth.

The above V then everywhere gives the r^{-1} term and the second harmonic exactly. For the fourth harmonic it yields an amplitude which has about half of Jeffreys' value[1] or about 86% of the value obtained by King-Hele and Merson,[2] from analysis of data on satellite orbits. Since these authors indicate a standard deviation of 16%, the resulting value for the fourth harmonic agrees with their value within the limits of observational error. Even the sixth harmonic, of amplitude 1.3×10^{-9}, falls within the (admittedly wide) amplitude limits, $(0.1\pm1.5)\times10^{-6}$, given by King-Hele and Merson.

[The term $b_1\,\mathrm{Im}(\xi+i\eta)^{-1}$ leads to odd harmonics, with amplitudes all proportional to the first harmonic amplitude δ/R, where R is the equatorial radius and δ is the distance OC from the coordinate center O to the center of mass C. Thus if O is taken as coincident with C, all the odd harmonics drop out, so that the potential $V(\xi,\eta)$ here derived always has symmetry with respect to a plane through C, perpendicular to the polar axis. It remains an open question whether use of a displaced center O, noncoincident with C, which would give rise to first, third, ..., harmonics with amplitudes δ/R, $0.00109\delta/R$, ..., could usefully represent equatorial asymmetry, as found by O'Keefe, Eckels, and Squires.[3]]

The net result of the present work is a reduction of the problem of satellite motion to quadratures, for a potential whose even harmonics are exact through the second, and whose fourth harmonic is comparable with empirical values. The kinetic equations of motion have been set up formally, in terms of certain integrals, which are now being evaluated. The effects of drag, of axial asymmetry, and of higher odd harmonics remain to be handled by perturbation theory.

This paper had its inspiration in the earlier work of Sterne[4] and of Garfinkel,[5] who also reduced the problem to quadratures, for potentials with approximate second harmonics.

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[1] Harold Jeffreys, The Earth (Cambridge University Press, Cambridge, 1952), second edition, Chap. IV.

[2] D. G. King-Hele and R. H. Merson, Nature 183, 881 (1959).

[3] O'Keefe, Eckels, and Squires, Science 129, 565 (1959).

[4] T. E. Sterne, Astron. J. 62, 96 (1957).

[5] B. Garfinkel, Astron. J. 63, 88 (1958).