GAUGE DEPENDENCE OF THE WAVE-FUNCTION RENORMALIZATION CONSTANT IN QUANTUM ELECTRODYNAMICS*

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We should like to point out the existence of an exact and simple relation between the electron Green's function renormalization constants in the general class of "manifestly" covariant gauges. We consider those gauges where the unrenormalized zeroth-order photon Green's function in momentum space has the form

$$D_{\mu\nu} = D_{\mu\nu}{}^0 - k_\mu k_\nu \lambda(k), \tag{1}$$

where λ is an arbitrary function of k. By $D_{\mu\nu}{}^0$ we shall mean a covariant Green's function in a fixed gauge.

One can show that the exact, unrenormalized electron Green's functions in the respective gauges are connected by the relation

$$G_\lambda(x-x') = \exp\{ie_0{}^2[\lambda(x-x') - \lambda(0)]\}G_0(x-x'), \tag{2}$$

where e_0 is the unrenormalized charge of the electron, and

$$\lambda(x) = \int \frac{(dk)}{(2\pi)^4} e^{ik\cdot x}\lambda(k). \tag{3}$$

This result has been derived previously,[1] but in a manner which seems to us somewhat questionable. An independent and rigorous derivation is possible using the techniques of external sources, and is given in papers on the gauge group in electrodynamics.[2,3] The authors of these latter papers were motivated by somewhat different considerations. It should also be marked that the connection between the Green's function in the radiation gauge [which does not fall into class (1)] and the Lorentz gauge has been derived[4] by an alternative method which should also apply to the case under consideration.

If the transformation is nonsingular, the function $\lambda(x)$ vanishes as $x^2 \to \infty$. Further, if the singularity of G at $p^2 = -m^2$ is made rigorously a pole by cutting off the soft photons with an invariant photon mass μ, then asymptotically in any gauge

$$G(x-x') \to Z_2 G_m(x-x') \tag{4}$$

as $(x-x')^2 \to \infty$, where

$$G_m(x-x') = \int \frac{(dp)}{(2\pi)^4} \frac{e^{ip\cdot(x-x')}}{\gamma\cdot p + m}, \tag{5}$$

since m, the physical mass of the electron, is the lowest invariant frequency contained in $G(x-x')$. Z_2 is the so-called wave-frequency renormalization constant. It follows from Eq. (2) that the wave-function renormalization constants in the two gauges are connected by the relation

$$Z_2{}^\gamma = \exp\{-ie_0{}^2\lambda(0)\}Z_2{}^0. \tag{6}$$

To give an example of this relation, one may take for $D_{\mu\nu}{}^0$ the function

$$D_{\mu\nu}{}^0 = \delta_{\mu\nu}\left(\frac{1}{k^2+\mu^2} - \frac{1}{k^2+\Lambda^2}\right), \tag{7}$$

where μ is the photon mass and Λ is inserted so that we may regard local electrodynamics as the limit of a cutoff theory. Then we choose for λ the function

$$\lambda = \gamma\left(\frac{1}{k^2+\mu^2} - \frac{1}{k^2+\Lambda^2}\right)\frac{1}{k^2}, \tag{8}$$

so that we study the one-parameter family of gauges for which

$$D_{\mu\nu} = \left(\delta_{\mu\nu} - \gamma\frac{k_\mu k_\nu}{k^2}\right)\left(\frac{1}{k^2+\mu^2} - \frac{1}{k^2+\Lambda^2}\right) \tag{9}$$

In this case, $\lambda(0) = i(\gamma/8\pi^2)\ln(\Lambda/\mu)$; therefore

$$Z_2{}^\gamma = (\Lambda/\mu)^{\gamma\alpha_0/2\pi} Z_2{}^0, \tag{10}$$

where $\alpha_0 = e_0{}^2/4\pi$ is the unrenormalized coupling constant.

If Λ is allowed to approach infinity, it is clear that $Z_2{}^\gamma$ can approach a finite and nonvanishing limit at most in one of the gauges of class (9). Thus, if as $\Lambda \to \infty$,

$$Z_2{}^0 \to (\Lambda/\mu)^{-\epsilon\alpha_0/2\pi} Z_2{}^f,$$

where Z_2^f is finite and not zero, then in the gauge in which $\gamma=\epsilon$ the renormalization constant would be finite and equal to Z_2^f in this limit. However, we wish to stress that even if such a special gauge did not exist, these considerations would in no way affect the possibility that the unrenormalized physical theory is consistent. Indeed, the reason that a simple relation of the type (2) exists is that the gauge change refers to the way in which the longitudinal and scalar quanta are coupled to the electron, and these quanta (because of the conservation of the fluctuating vacuum currents) enter in an essentially noninteracting way. The factor $(\Lambda/\mu)^{\gamma\alpha_0/2\pi}$ is produced by the change in the number of these quanta in the state in which $p^2=-m^2$, and consequently has nothing to do with the physical theory, which is concerned only with the transverse quanta. Therefore, the fact that these degrees of freedom introduce inconsistencies only indicates that the use of formulations of electrodynamics that employ them is dangerous in the investigation of the consistency of the (unrenormalized) physical theory at high energies. This points to the radiation gauge as useful for these purposes since it does not fall into class (1), unphysical quanta are avoided, and the Green's functions are directly related to matrix elements of operators in a Hilbert space with a physically acceptable (positive) metric.

In spite of these difficulties, which have to do with the consistency of the local theory, one may still make use of the gauges of class (9) in a pragmatic way provided that Λ is chosen larger than any energy of physical interest. In this case it is useful to note that information about the infrared structure of the Green's function in various gauges may be derived from Eq. (2). It should be stressed again that changing the gauge alters only the number of scalar and longitudinal quanta in the various states and hence has nothing to do with physics. However, simplifications in calculations may occur if the gauge is chosen properly. In the low-energy domain (as opposed to the infinite-energy domain) the transverse quanta should enter in an essentially noninteracting way once the charge is renormalized. It is therefore not inconceivable that cancellations between real quanta and the longitudinal and scalar quanta could be arranged so that a gauge of type (9) might exist where G has rigorously a pole at $p^2=-m^2$, in the limit as $\mu\to 0$. In this case, the singularity in any other gauge can be calculated quite simply by using (2).

If we allow $\mu\to 0$, the difference $ie_0^2[\lambda(x)-\lambda(0)]$ remains finite and becomes equal to

$$-\frac{i\gamma\alpha_0}{2}\left\{\frac{H_1^{(2)}(\Lambda(-x^2)^{1/2})}{\Lambda(-x^2)^{1/2}}+\frac{i}{\pi}\ln[\tfrac{1}{2}\gamma\Lambda(-x^2)^{1/2}]-\frac{i}{2\pi}-\frac{1}{2}\right\}.$$

Therefore, for $\Lambda(-x^2)^{1/2}\gg 1$,

$$ie_0^2[\lambda(x)-\lambda(0)]=(\gamma\alpha_0/2\pi)\ln[m(-x^2)^{1/2}]+\text{const.} \quad (11)$$

Suppose now that there exists a gauge with $\gamma=\bar{\gamma}$, where the Green's function has a pole at $p^2=-m^2$. Then it is possible to show that in the gauge corresponding to an arbitrary value of γ, the Green's function in the neighborhood of $p^2=-m^2$ will behave like

$$\frac{1}{\gamma\cdot p+m}\left\{\frac{m^2}{p^2+m^2}\right\}^{\alpha_0(\gamma-\bar{\gamma})/2\pi} \quad (12)$$

The work of Yennie[5] indicates that such a gauge exists with

$$\bar{\gamma}=1-3(\alpha/\alpha_0), \quad (13)$$

where α is the renormalized coupling constant. This is explicitly verified to the fourth order by Solov'ev.[6]

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