FIELD THEORY COMMUTATORS

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There are some paradoxical contradictions between the formal commutation relations and the nature of the energy spectrum which serve to emphasize that localized field operator products must be understood as the singular limit of products defined for noncoincident points. Thus, it is customary to assert that the electric charge density of a Dirac field commutes with the current density at equal times, since the current vector is a gauge-invariant bilinear combination of the Dirac fields. It follows from the conservation of charge that the charge density and its time derivative, referring to any pair of spatial points at a common time, are commutative. But this is impossible if a lowest energy state—the vacuum—is to exist. For any Hermitian operator F, here an arbitrary linear functional of the charge density, the vacuum expectation value of such a commutator is

$$\langle [i\partial_0 F, F]\rangle = \langle [[F, P^0], F]\rangle = 2\langle FP^0F\rangle > 0,$$

since the operator F in general will possess nonvanishing matrix elements between the vacuum state of zero energy and other states of necessarily positive energy. The exceptional circumstance, that the vacuum state is an eigenvector of F, is excluded physically for the example of the charge flux vector. A related example appears on deriving F from a component of the electric field by choosing the zero wave number Fourier transform. It is the advantage of this special choice that the time derivative of F is then proportional to the same Fourier component of the electric current vector, which enables us to conclude that the electric field and the current vector cannot commute. Invariance considerations and an existence hypothesis show that

$$x^0 = x^{0\prime}: \quad \langle i[E_k(x), j_l(x')]\rangle = \delta_{kl}\delta(\vec{x}-\vec{x}')K^2,$$

where K is a real constant with the dimensions of inverse length or mass. Since the longitudinal part of the electric field is related to the charge density, we learn that

$$\langle i[j^0(x), j_l(x')]\rangle = \nabla_l \delta(\vec{x}-\vec{x}')K^2.$$

The independent nonvanishing commutator expectation value involving the transverse electric field is also in apparent contradiction with the formal commutation properties of the Maxwell and Dirac fields.

Electromagnetic field commutation relations inferred from invariance requirements must be realized by any model of the charge-bearing field. For a spinless field $\phi(x)$, $\phi^\dagger(x)$,

$$j_k = \tfrac{1}{2}e[\phi^\dagger\left(\frac{1}{i}\nabla_k - eA_k\right)\phi + (i\nabla_k - eA_k)\phi^\dagger\phi],$$

in which symmetrization of the $\phi^\dagger\phi$ products is understood, and the application of the formal commutation relations now yields the anticipated result, with

$$K^2 = e^2\langle\phi^\dagger\phi\rangle.$$

The electric current vector of the Dirac field is generally defined as the limit of a time-ordered product,

$$j_\mu(x) = \lim_{x'\to x} e(\psi^\dagger(x)\gamma^0\gamma_\mu\psi(x'))_+ \epsilon_+(x-x'),$$

but for the purpose of evaluating the commutator with the charge density, it suffices to consider the operator product at distinct spatial points[1] and equal times, as in

$$i[\int(d\vec{x})\phi(x)j^0(x), e\psi^\dagger(x'-\tfrac{1}{2}\vec{\epsilon})\gamma^0\gamma_k\psi(x'+\tfrac{1}{2}\vec{\epsilon})]$$
$$= -(\phi(x'+\tfrac{1}{2}\vec{\epsilon}) - \phi(x'-\tfrac{1}{2}\vec{\epsilon}))ie^2\psi^\dagger(x'-\tfrac{1}{2}\vec{\epsilon})\gamma^0\gamma_k\psi(x'+\tfrac{1}{2}\vec{\epsilon}).$$

If the limit $\vec{\epsilon}\to 0$ is performed symmetrically in space, the required expectation value form is obtained, with

$$K^2 = -\tfrac{1}{3}ie^2\lim_{\vec{\epsilon}\to 0}\mathrm{tr}\,\gamma^0\vec{\gamma}\cdot\vec{\epsilon}\langle\psi(x+\tfrac{1}{2}\vec{\epsilon})\psi^\dagger(x-\tfrac{1}{2}\vec{\epsilon})\rangle.$$

The conventional treatment of the commutator evidently assumes that the field product expectation value remains bounded as $\vec{\epsilon}\to 0$, whereas the structure of the energy spectrum provides assurance to the contrary. In one simple situation, that of the noninteracting Dirac field, it is well known that

$$\langle\psi(x+\tfrac{1}{2}\vec{\epsilon})\psi^\dagger(x-\tfrac{1}{2}\vec{\epsilon})\rangle \sim \frac{1}{2\pi^2}\frac{i\gamma^0\vec{\gamma}\cdot\vec{\epsilon}}{(\vec{\epsilon}^2)^2}, \quad \vec{\epsilon}\to 0$$

and K^2 is the divergent limit of $(2e^2/3\pi^2)(\vec{\epsilon}^2)^{-1}$.

The implications of this discussion for the structure of the photon Green's function will be considered elsewhere.

[1]An additional factor involving the line integral of the vector potential should be understood. It is needed to maintain gauge invariance and accounts for the nonvanishing commutator with the transverse electric field.

K^+ - K^0 MASS DIFFERENCE

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It has recently been established[1] that, in contrast to the π mesons, the K^0 is heavier than the $K^\pm$. This is in contradiction with perturbation theory,[2] and with the naive classical argument that the charged components of a mass multiplet should be heavier in proportion to the work done in assembling the charge. We wish to draw attention to the important difference that, whereas the π^0 has a vanishing charge density, the K^0, like the neutron, is a concentration of positive and negative charge,[3] giving total charge zero, but with a nonvanishing charge density. It is quite simple to construct physically reasonable charge distributions for K^0 and K^+ such that the neutral particle has the greater electrostatic energy. We produce such a model and show further that this classical interpretation of the self-energy is very closely related to the leading term in the field-theoretic calculation.

The familiar relation[4] between electric charge, isotopic spin, I, and hypercharge, Y, can be expressed in terms of the corresponding current densities, in an obvious notation, as

$$j_\mu^{el}(x) = j_\mu^{I_3}(x) + \tfrac{1}{2} j_\mu^{Y}(x). \tag{1}$$

The static charge distribution for a particular multiplet, χ, is the expectation value of the "zero" component of this operator for single-particle states in the rest system:

$$\rho(x) = \langle \chi | j_0^{el}(x) | \chi \rangle = I_3 \rho_I(x) + \tfrac{1}{2} Y \rho_Y(x), \tag{2}$$

where I_3 is the matrix appropriate to the multiplet and Y is the single-particle value.

For π^0 it follows that

$$\rho(x) = 0. \tag{3}$$

The neutral pion has a vanishing charge density, and the fact that the charged pions are heavier is consistent with the simple classical argument.[5]

For the case of the K meson ($Y=1$) the situation is quite different. Rewriting Eq. (2),

$$\rho_K(x) = \tfrac{1}{2}(1+\tau_3)\rho_1(x) + \rho_2(x), \tag{4}$$

it is seen that ρ_2 describes the charge density of the neutral K meson. We argue that by choosing a suitable charge distribution, say a negative "cloud" around a sufficiently small positive "core", one can easily give the neutral K meson a greater self-mass than the charged one. Choose, for example,

$$\rho_+(x) = (e^2/4\pi)\lambda^2 e^{-\lambda r}/r = \rho_1 + \rho_2, \tag{5a}$$

$$\rho_0(x) = (e^2/4\pi)[\lambda^2 e^{-\lambda r} - \mu^2 e^{-\mu r}]/r = \rho_2. \tag{5b}$$

Recalling that the classical self-mass is given (for the charged particle for example) by

$$\delta M_+ = \frac{e^2}{(2\pi)^2}\int d\vec{q}\, \frac{\lambda^2}{\lambda^2+\vec{q}^2}\, \frac{1}{k^2}\, \frac{\lambda^2}{\lambda^2+\vec{q}^2}, \tag{6}$$

it is found that the observed mass splitting is reproduced by taking λ to be a pion Compton wavelength and μ, the "core" radius, a little greater than two Σ-hyperon Compton wavelengths.

In the field-theoretic calculation, it is reasonable to suppose that the main contribution comes from intermediate states of the least possible energy, and this is borne out by the dispersion relation approach.[6] We need to calculate the contribution from the graph of Fig. 1. The cor-

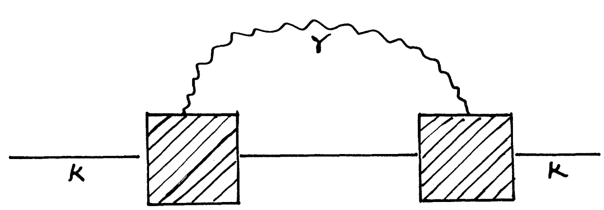

FIG. 1. Feynman diagram of the K-meson electromagnetic self-energy.