MODIFICATION OF THE EFFECTIVE-RANGE FORMULA FOR NUCLEON-NUCLEON SCATTERING*

H. Pierre Noyes and David Y. Wong
Lawrence Radiation Laboratory, University of California, Berkeley and Livermore, California
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The effective-range formula is well known to be a useful description of low-energy scattering phase shifts. However, except at the limit of zero kinetic energy, the effective-range formula is, in general, not exact. We report here a study of the deviation from the effective-range formula in light of the analytic structure of partial-wave amplitudes as suggested by Mandelstam.[1] The problem under consideration is the s-wave nucleon-nucleon scattering in both the singlet and the triplet states.

The effective-range formula, as it stands, implies that the only singularities of the s-wave amplitude are two poles in the complex momentum plane:

$$\frac{1}{q}e^{i\delta}\sin\delta = 1\bigg/\left(-\frac{1}{a}+\frac{r}{2}q^2 - iq\right) = \frac{(q_1+q_2)}{i(q-q_1)(q-q_2)}, \quad (1)$$

where

$$q_1 = i\left[\frac{1}{r}+\left(\frac{1}{r^2}-\frac{2}{ar}\right)^{\frac{1}{2}}\right], \quad q_2 = i\left[\frac{1}{r}-\left(\frac{1}{r^2}-\frac{2}{ar}\right)^{\frac{1}{2}}\right]. \quad (2)$$

Here q is the center-of-mass momentum, a is the scattering length, and r is the effective range. The effective-range expression (1) can equivalently be characterized by q_1 and the residue at q_1:

$$\Gamma_1 = \frac{1}{i}\left(\frac{q_1+q_2}{q_1-q_2}\right), \quad (3)$$

where q_1 is chosen rather than q_2 because it has a closer relation to the "interaction." In fact, if $(i\Gamma_1)$ is sufficiently large, q_2 will become a bound-state pole. This connection between the triplet s-wave scattering amplitude and the deuteron bound state is well known. We note also that, as long as the effective range is positive, q_1 is always on the upper half-plane.

According to Mandelstam's representation, partial-wave amplitudes for nucleon-nucleon scattering are analytic on the upper-half q plane except for branch points on the positive imaginary axis corresponding to thresholds for one-, two-, and three-meson exchange, etc. Bound-state poles will appear as the "strength" of the branch points becomes sufficiently great. In view of the analyticity implied by Mandelstam's representation, the "interaction pole" (q_1) in the effective-range formula can be considered as an approximate replacement for the branch cuts predicted by field theory. It is evident that an improvement over the effective-range formula can be obtained by including the one-meson branch cut exactly, and allowing the pole to represent only the average contribution of the remaining cuts which are farther away from the physical region (the one-, two-, three-, $\cdots$-meson branch points are located at $q=i/2$, i, $3i/2$, $\cdots$, respectively). We shall construct such a function in the following paragraph.

It is convenient, at this point, to introduce the momentum-square variable, $\nu = q^2$. The s-wave amplitude can be written as

$$h(\nu) = \frac{1}{\nu^{1/2}\cot\delta(\nu) - i\nu^{1/2}}.$$

The unitarity condition implies that the inverse function $h^{-1}(\nu)$ has a branch point at $\nu=0$ with a discontinuity across the cut from 0 to ∞ given by $-2i\sqrt{\nu}$. The one-meson cut for $h(\nu)$ can be calculated exactly in terms of the renormalized pion-nucleon coupling constant f^2. The discontinuity across the one-meson cut is simply $(\pi i f^2 M/2\nu)$ for $-\infty \leq \nu \leq -\frac{1}{4}$. M is the nucleon mass in pion units. This same cut holds for both the singlet and the triplet s-wave amplitudes. The mixing of d wave in the triplet state is neglected in this calculation. An approximate calculation of the s-d mixing has already been reported by one of us (DYW).[2] For the construction of a function with a given branch cut along the negative real axis and a given branch cut for the inverse function along the positive real axis, we express $h(\nu)$ in the form of a quotient,[3]

$$h(\nu) = N(\nu)/D(\nu). \quad (5)$$

Letting $N(\nu)$ be analytic except for a branch point at $-\frac{1}{4}$ and a pole at ν_1, and $D(\nu)$ be analytic except for a branch point at the origin, one can immediately arrive at the coupled integral equations

$$N(\nu) = \frac{\Gamma}{\nu-\nu_1} + \frac{f^2M}{4}\int_{-\infty}^{-1/4} d\nu' \frac{D(\nu')}{\nu'(\nu'-\nu)}, \quad (6)$$

$$D(\nu) = 1 - \frac{(\nu-\nu_1)}{\pi}\int_0^{\infty} d\nu' \frac{(\nu')^{\frac{1}{2}}N(\nu')}{(\nu'-\nu_1)(\nu'-\nu)}. \quad (7)$$

The subtraction is made in Eq. (7) to keep the consistent asymptotic behavior of $h(\nu) \to O[(1/\nu)\ln\nu]$

for $f^2 \neq 0$, and $h(\nu) \to O(1/\nu)$ for $f^2 = 0$. In the limit $f^2 = 0$, the solution of Eqs. (6) and (7) reduces to the effective-range formula (1). The residue Γ is then trivially related to Γ_1 of Eq. (3). For a coupling constant greater than zero, Eqs. (6) and (7) can be solved by a straightforward iteration in (f^2M). The series is uniformly convergent for $(f^2M) \leq 1$ (the actual radius of convergence may be greater than 1). In this calculation we use $f^2 = 0.08$, $(f^2M) = 0.533$; Γ and ν_1 are adjusted to fit two precisely known singlet S p-p phase shifts at 1.397 and 2.425 Mev,[4] or the deuteron binding energy and the triplet scattering length. Calculated $\nu^{1/2}\cot\delta$ curves are given in Figs. 1 and 2 for the singlet p-p and the triplet n-p, respectively.[5] Shape parameters have also been calculated, and the quadratic approximation

$$\nu^{1/2}\cot\delta = -1/a + \tfrac{1}{2}r\nu - Pr^3\nu^2, \tag{8}$$

is plotted on the corresponding figures.

We summarize our results as follows:

(a) Since the effective "interaction pole" replaces only those branch points at $\nu \leq -1$, the present result should be reasonably reliable up to $\nu \leq 1$ ($\cong$40 Mev).

(b) Although the power series in ν diverges for $|\nu| \geq \frac{1}{4}$ because of the one-meson branch point, our calculated $\nu^{1/2}\cot\delta$ curve remains quite close to a straight line up to $\nu = 1$.

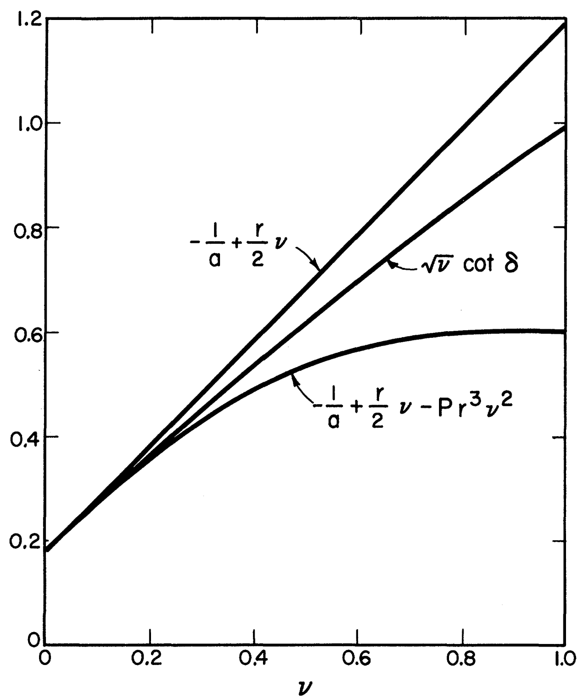

FIG. 1. Singlet p-p scattering, with $a = -5.5$, $r = 2.0$ ($\nu_1 = -2.6$, $P = 0.073$).

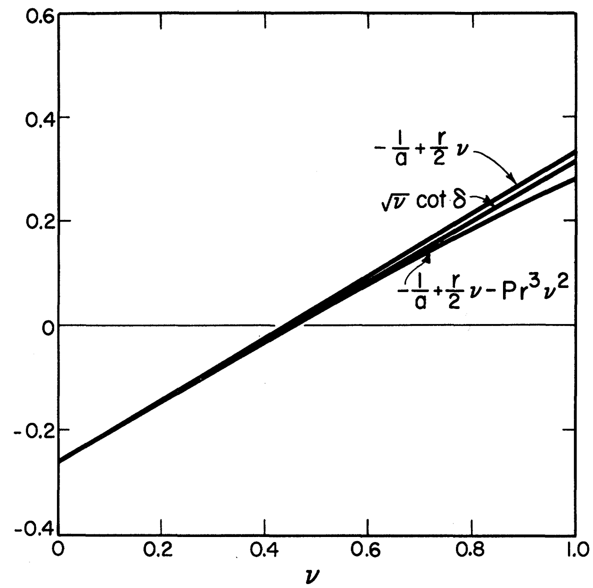

FIG. 2. Triplet n-p scattering, with $a = 3.8$, $r = 1.2$ ($\nu_1 = -2.0$, $P = 0.028$).

(c) As f^2 goes from zero to 0.08, ν_1 moves from -1.8 to -2.0 for the triplet amplitude and from -1.34 to -2.6 for the singlet p-p amplitude. The ratio $(-\Gamma/\nu_1)$, which is an approximate measure of the effect of the pole, decreases by 14% for the triplet and 30% for the singlet. These numerical results point to the fact that the one-meson force is far from sufficient to give the required attraction.[6] However, the qualitatively reasonable positions and strengths of these "interaction poles" have added to our confidence in replacing unknown branch cuts by such poles. Also, it indicates that the successive inclusion of outer branch cuts would make the role of the phenomenological "interaction pole" less and less important. A systematic procedure of successive approximations appears to be quite possible.

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[1]S. Mandelstam, Phys. Rev. 112, 1344 (1958). Mandelstam has proved his representation through the sixth order in perturbation theory. J. Bowcock and D. Walecka (private communication) have proved the fixed-angle dispersion relation for the Born series of nonrelativistic potential scattering with Yukawa-type potentials. Independently, Blankenbecler, Goldberger, Khuri,

and Trieman have proved the Mandelstam representation for the same potential scattering problem to all orders in the Born series as well as the Fredholm series (private communication).

[2]D. Y. Wong, Phys. Rev. Letters 2, 406 (1959).

[3]This method was suggested by G. F. Chew and S. Mandelstam, Lawrence Radiation Laboratory Report UCRL-8728, April 1, 1959 (unpublished), in connection with an analogous problem for pion-pion interaction.

[4]MacGregor, Moravcsik, and Noyes, University of California Radiation Laboratory Report UCRL-5582-T; Knecht, Messelt, Berners, and Northcliffe, Phys. Rev. 114, 550 (1959).

[5]In making use of the two proton-proton phase shifts, we have simply assumed that $\nu^{1/2}\cot\delta$ as calculated from Eqs. (6) and (7) is to be compared with the usual Coulomb modification

$$\{[2\pi nq\cot\delta/\exp(2\pi n)-1]+2qnh(q)\};\; n=e^2/\hbar v.$$

[6]Cini, Fubini, and Stanghellini; W. Alles and A. Tomasini; and S. Matsuyama (private communication) have made separate attempts to determine the coupling constant from dispersion relations using observed s-wave parameters. It is clear that at least three parameters are needed for a reliable determination of the coupling constant. It seems to us that our present knowledge of the s wave is inadequate to give such a three-parameter set. However (Fubini and Stanghellini, private communication) if one assumes the value of f^2 known, the formulas for $q\cot\delta$ and its derivative as given by Cini, Fubini, and Stanghellini evaluated at $q^2=-1/2$ imply a positive shape parameter and a $^1S_0$ phase shift of $\sim 48°$ at 40 Mev, as do our formulas. Preliminary results indicate that experiment may support this conclusion [MacGregor, Moravcsik and Noyes, University of California Radiation Laboratory Report-UCRL-5582-T (unpublished)].

ERRATUM

EVIDENCE FOR ANISOTROPY OF THE SUPERCONDUCTING ENERGY GAP FROM ULTRASONIC ATTENUATION. R. W. Morse, T. Olsen, and J. D. Gavenda [Phys. Rev. Letters 3, 15 (1959)].

Measurements reported in this paper for propagation along the [100] direction have been found to be erroneously labelled. Actually these data were taken for a direction of propagation which was perpendicular to [001] and 18° from [100]. From subsequent measurements along [100] one estimates the limiting energy gap to be $(3.5\pm 0.1)kT_C$ for this direction.