VARIATIONAL PRINCIPLE FOR GEOMETRODYNAMICS

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The purpose of this note is to propose a variational principle for geometrodynamics, to derive from it certain of the field equations, and to relate the problem of obtaining the remaining field equations to the problem of finding a superpotential for the metric tensor.

Misner and Wheeler,[1] following Rainich,[2] have shown that the content of general relativity and classical source-free electromagnetism can be fully expressed in terms of quantities depending only on the metric tensor $g_{\mu\nu}$ of a Riemannian space. In their formulation the field equations, fully equivalent to the usual ones, are

$$R_{\mu\alpha}R^{\alpha}{}_{\nu}=g_{\mu\nu}(R_{\alpha\beta}R^{\alpha\beta})/4, \tag{1}$$

$$R_{\mu}{}^{\mu}=R=0, \tag{2}$$

$$R_{00}\geqslant 0, \tag{3}$$

$$\alpha_{\beta|\gamma}-\alpha_{\gamma|\beta}=0, \tag{4}$$

where the vector α_β is defined as

$$\alpha_{\beta}=(-g)^{1/2}[\beta\lambda\mu\nu]R^{\lambda\gamma|\mu}R_{\gamma}{}^{\nu}/R_{\sigma\tau}R^{\sigma\tau}. \tag{5}$$

For some purposes it is useful to replace the $10+1=11$ equations (1) and (2) by the equivalent 10 equations:

$$\text{“}-\tfrac{1}{2}\text{”}g_{\mu\nu}R-\frac{R_{\mu\alpha}R^{\alpha}{}_{\nu}}{(R_{\sigma\tau}R^{\sigma\tau})^{1/2}}+\tfrac{1}{4}g_{\mu\nu}(R_{\sigma\tau}R^{\sigma\tau})^{1/2}=0. \tag{6}$$

The identity of (6) to (1) and (2) may be seen by setting $\mu=\nu$ and summing. The factor $-\frac{1}{2}$ could just as well have any other constant value—hence the quotes.

By way of explanation it may be recalled that the curl condition (4) guarantees the existence of a scalar,

$$\alpha(x)=\int_0^x \alpha_\beta dx^\beta+\alpha_0, \tag{7}$$

the "complexion" of the electromagnetic field. If $\alpha=n\pi$ $(n=0,1,2,\ldots)$, we have a pure electric field; if $\alpha=\frac{1}{2}n\pi$ $(n=1,3,\ldots)$, the field is pure magnetic. The complexion at a point, together with the value of the Ricci curvature tensor at the point, completely determine the electromagnetic field $f_{\mu\nu}$, and hence also the stress-energy tensor,

$$T_{\mu\nu}=f_{\mu\alpha}f^{\alpha}{}_{\nu}-\tfrac{1}{4}g_{\mu\nu}f_{\sigma\tau}f^{\sigma\tau}, \tag{8}$$

at that point.

The algebraic equations (1)-(3) satisfied by the Ricci curvature express the two-way connection between the electromagnetic field tensor $f_{\mu\nu}$ and the Ricci tensor $R_{\mu\nu}$. Thus, any electromagnetic field tensor $f_{\mu\nu}$ produces a Ricci curvature $R_{\mu\nu}$ that satisfies (1)-(3). Conversely, given an $R_{\mu\nu}$ that satisfies (1)-(3), we can solve for $f_{\mu\nu}$, the solution being uniquely determined except for α. The curl condition (4) subsequently determines α up to an additive constant α_0.

It is often very helpful to have a variational formulation of a set of field equations.

Such a formulation is well known[3,4] in the more familiar version of general relativity plus electromagnetism, where the $f_{\mu\nu}$ are treated as foreign objects immersed in the metric space rather than expressed—as above in Eqs. (1)-(5)—directly in purely geometric terms.

In the traditional version of general relativity plus electromagnetism, the coupled field equations are

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=f_{\mu\alpha}f^{\alpha}{}_{\nu}-\tfrac{1}{4}g_{\mu\nu}f_{\sigma\tau}f^{\sigma\tau}, \tag{9}$$

$$f^{\mu\nu}{}_{|\nu}=0, \tag{10}$$

$${}^{*}f^{\mu\nu}{}_{|\nu}=0. \tag{11}$$

The variational formulation of these equations has the form

$$\delta I=0,\quad I=\int \mathfrak{L}\,d\tau,\quad d\tau=dx^0dx^1dx^2dx^3, \tag{12}$$

with

$$\mathfrak{L}=(\text{Lagrangian density for the gravitational field}) + (\text{Lagrangian density for electromagnetic field}) \tag{13}$$

$$=R(-g)^{1/2}+\tfrac{1}{2}(-g)^{1/2}f_{\sigma\tau}f^{\sigma\tau}. \tag{14}$$

The usual procedure to obtain the field equations from the above is to require (for detail, see references 3 and 4):

(a) The existence of a vector 4-potential for $f_{\mu\nu}$:

$$f_{\mu\nu}=\varphi_{\mu|\nu}-\varphi_{\nu|\mu} \rightleftharpoons f^{\mu\nu}{}_{|\nu}=0; \qquad (15)$$

(b) $f_{\mu\nu}$ (or φ_μ) and $g^{\mu\nu}$ are treated as the quantities to be varied;

(c) $\delta\varphi_\mu$ and $\delta g^{\mu\nu}$ are both to vanish on the surface σ bounding τ.

We propose for geometrodynamics the Lagrangian density

$$\mathcal{L}=R(-g)^{1/2}-\tfrac{1}{2}(-g)^{1/2}(R_{\mu\nu}R^{\mu\nu})^{1/2} \times\cos2\left[\int_0^x \alpha_\beta dx^\beta\right]. \qquad (16)$$

The first term is, as before, the Lagrangian density for the gravitational field. Likewise, the second term is an expression for the Lagrangian density E^2-H^2 for the electromagnetic field, expressed in purely geometrical form. This expression can be derived from results in Misner and Wheeler.[5]

We see that our Lagrangian density is a functional of the metric tensor $g^{\mu\nu}$ and its derivatives up to the second order, and of the path of integration π occurring in the definition of $\alpha(x)$, Eq. (7).

We require that $I=\int\mathcal{L}d\tau$ be stationary with respect to arbitrary variations of both the path π and $g^{\mu\nu}$.

We first vary with respect to π. $\delta_\pi I=0$ results in the condition that the integral

$$\int_0^x \alpha_\beta dx^\beta, \qquad (17)$$

be independent of the path from 0 to x. This will be true if the integrand α_β has a vanishing curl:

$$\alpha_{\beta|\gamma}-\alpha_{\gamma|\beta}=0. \qquad (4)$$

This is one of the equations we seek. We may now write $\mathcal{L}$ in a slightly simpler form:

$$\mathcal{L}=R(-g)^{1/2}-\tfrac{1}{2}(R_{\mu\nu}R^{\mu\nu})^{1/2}\cos2\alpha. \qquad (18)$$

With no loss of generality we can set $\alpha(x)=0$ at the point of variation, since this amounts to nothing more than choosing a convenient origin of coordinates. Note, however, that this does not imply that $\delta\alpha=0$, although $\delta(\cos2\alpha)=0$.

At this point it would seem natural (1) to vary Eq. (18) with respect to $g^{\mu\nu}$, (2) to integrate by parts terms containing derivatives of $\delta g^{\mu\nu}$, and (3) to set equal to zero the resulting coefficient of $\delta g^{\mu\nu}$. The field equations obtained in this way are not satisfactory; they are not equivalent to the desired and expected Eq. (6).

It seems reasonable to attribute this difficulty to a wrong choice of the quantities to hold fixed on the surface, not to a wrong choice of quantities to be varied.

This is not the first time that such a problem has arisen, as witness the variational principle:

$$\delta\int(E^2-H^2)d(\mathrm{vol})=\delta\int(-g)^{1/2}f_{\mu\nu}g^{\mu\alpha}g^{\nu\beta}f_{\alpha\beta}d\tau=0, \qquad (19)$$

for electromagnetism alone in a space with prescribed metric $g_{\mu\nu}$.

The most primitive analysis of this variational principle, based on the assumption that the $f_{\mu\nu}$ are the quantities to be varied, and that the variations of $f_{\mu\nu}$ vanish at the boundaries, leads to the incorrect result that $f_{\mu\nu}=0$ throughout the interior.

The customary procedure is to add a supplementary condition, Eq. (15), and treat the φ_μ as the quantities to be varied, and hold them fixed on the boundary.

By setting the coefficient of $\delta\varphi_\mu$ equal to zero, we obtain four of Maxwell's equations. The other four were assumed from the outside, Eq. (15).

However, there is still a third approach,[6-9] due to Finzi, whereby one can obtain all eight of Maxwell's equations without the addition of such a subsidiary condition from outside.

Finzi begins by observing that any antisymmetric tensor $f_{\mu\nu}$ can be expressed in the form

$$f_{\mu\nu}=A_{[\mu|\nu]}+\frac{(-g)^{1/2}[\mu\nu\alpha\beta]g^{\mu\alpha}g^{\nu\beta}B_{[\alpha|\beta]}}{2}, \qquad (20)$$

where A_μ and B_α are 4-vectors. We now vary the action integral with respect to both A_μ and B_α, subject to the condition that they both be fixed on the boundary. The demand that the action be stationary immediately produces all eight of Maxwell's equations.

Judging by the analogy between the $f_{\mu\nu}$ as field variables in electromagnetism and the $g^{\mu\nu}$ as field variables in gravitation, we expect that there are alternative ways to vary arbitrarily the field variables within a space-time region. The ways differ from each other in the choice of quantities kept fixed on the 3-surface bounding this region.

It was wrong in electromagnetism to think of the $f_{\mu\nu}$ as fixed on the surface. It appears equally wrong in geometrodynamics to think of the $g^{\mu\nu}$ as fixed on the surface.

In electromagnetism we know the potentials which should be fixed on the surface. In geometrodynamics we do not yet know these potentials.

Therefore, we regard the problem ahead as not to change the variational principle Eq. (16), which is so plausible on physical grounds, but rather to find the right superpotential in terms of which to express the $g^{\mu\nu}$.

In discussions (1) of the quantization of general relativity and (2) of the "true variables" of general relativity, the hope has often been expressed to find these superpotentials. We now have a third and perhaps even stronger motive to construct these quantities: to validate—if possible—a variational principle which, for the first time, summarizes in purely geometrical terms the whole content of classical source-free electromagnetism and general relativity.

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POSSIBLE METHODS FOR THE DETERMINATION OF STRANGE-PARTICLE PARITIES*

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The question of the intrinsic parities of the strange particles has been extensively discussed in the past. The aim of this note is to point out two possible experiments that may shed light on this matter.

We first consider the capture reaction

$$K^- + d \rightarrow Y + N. \qquad (1)$$

It has been recently suggested that the capture of K^- in liquid hydrogen and deuterium occurs predominantly from bound s states.[1] In this note we shall consider that statement as a working hypothesis and shall derive a theorem that may be used to obtain information regarding the parity of the K meson relative to the YN system. With the assumption that K capture occurs predominantly from s states (bound or from the continuum) relative to the center of mass of the deuteron, the most general form of the transformation matrix in spin space is

$$M = [a + b(\vec{\sigma}_1 \cdot \vec{k})(\vec{\sigma}_2 \cdot \vec{k})]T \text{ for scalar } K, \qquad (2a)$$

and

$$M = [c\vec{\sigma}_1 \cdot \vec{k} + d\vec{\sigma}_2 \cdot \vec{k}]T \text{ for pseudoscalar } K, \qquad (2b)$$

where $\vec{k}$ is a unit vector along the direction of the relative momentum of the final system and T is the triplet projection operator. The operator T simply describes the spin correlation of the initial state. We also note that the other possible invariants $\vec{\sigma}_1 \cdot \vec{\sigma}_2 T$ and $\vec{\sigma}_1 \cdot (\vec{k} \times \vec{\sigma}_2)T$ can be reduced to the general forms given in Eqs. (2a) and (2b), respectively.

We now evaluate the correlation between the components of the spins of the Y and N along a direction $\vec{n}$, perpendicular to $\vec{k}$:

$$\langle \sigma_{1n}\sigma_{2n} \rangle = \mathrm{Tr}[MM^\dagger \sigma_{1n}\sigma_{2n}]/\mathrm{Tr}[MM^\dagger]. \qquad (3)$$

Inserting expressions (2a) and (2b) into (3), we obtain

$$\langle \sigma_{1n}\sigma_{2n} \rangle = |a-b|^2/[|a+b|^2 + 2(|a|^2 + |b|^2)] \geq 0, \quad (4a)$$

and

$$\langle \sigma_{1n}\sigma_{2n} \rangle = -|c-d|^2/[|c+d|^2 + 2(|c|^2 + |d|^2)] \leq 0, \quad (4b)$$