Transverse Diffusion and Conductivity Coefficients for a Three-Dimensional Magnetized Equilibrium Plasma*

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(Received 3 May 1972)

Equilibrium transport coefficients are calculated for a three-dimensional magnetized plasma. It is shown that the transverse spatial diffusion and conductivity coefficients are related via a generalized Einstein relation. These coefficients have both a $1/B$ (volume dependent) term (with B a uniform magnetic field) as well as a $1/B^2$ (volume independent) term for a finite plasma. For an infinite plasma, the coefficients have a "classical" $1/B^2$ dependence.

Recently there has been considerable interest in the two-dimensional (2D) guiding-center plasma.[1-4] In this model the particles are charged rods, aligned parallel to a constant uniform magnetic field $\vec{B}=B\hat{b}$ in the z direction, and interacting electrostatically through the logarithmic Coulomb potential. The guiding-center approximation implies that the velocity of each rod is given by $c\vec{E}\times\vec{B}/B^2$, where $\vec{E}$ is the electric field due to all the other particles. Anomalous $(1/B)$ equilibrium transport coefficients have been derived for this 2D plasma, in particular the spatial diffusion coefficient[1] and the conductivity coefficient,[4] both of which exhibited a volume-dependence divergence. In this Letter we calculate equilibrium transport (diffusion and conductivity) coefficients for a 3D plasma in which the particles move with the $\vec{E}\times\vec{B}$ drift *across* the field lines but whose motion *along* the field lines is given by Newton's laws of motion (i.e., we have generalized a 3D model used previously[5] in calculating the diffusion coefficient; in this model[5] the particles free streamed along the field lines, but here we are taking into account the effect of the fluctuating electric field on the particle motion along the field lines).

Consider a plasma of N electrons and N ions in a volume V with a uniform magnetic field $\vec{B}=B\hat{b}$ in the z direction. The fluctuating electric field is given by

$$\nabla\cdot\vec{E}(\vec{x},t)=4\pi\textstyle\sum_j e_j\delta(\vec{x}-\vec{R}_j(t)), \tag{1}$$

where $\vec{R}_j(t)$ is the position of the jth particle of mass m_j and e_j at time t, and the $\sum_j$ is a sum over all particles (electrons and ions).

The transverse spatial diffusion of a group of "test" ions, present in the electric field of Eq. (1), is given by

$$D_\perp=\int_0^\infty\langle\vec{v}_\perp(0)\cdot\vec{v}_\perp(t)\rangle dt=(c^2/B^2)\int_0^\infty\langle\vec{E}_\perp(0)\cdot\vec{E}_\perp(t)\rangle dt. \tag{2}$$

In calculating $D_\perp$ it is convenient to consider the electric-field autocorrelation tensor $\langle\vec{E}(0)\vec{E}(t)\rangle$, which on taking the Fourier transform of Eq. (1), is

$$\langle\vec{E}(0)\vec{E}(t)\rangle=\langle\vec{E}(0,0)\vec{E}(\vec{r}(t),t)\rangle=\sum_{ij}\frac{16\pi^2e_ie_j}{V^2}\sum_{\vec{k}}\frac{\vec{k}\vec{k}}{k^4}\langle\exp\{i\vec{k}\cdot[\vec{r}(t)+\vec{R}_i(0)-\vec{R}_j(t)]\}\rangle, \tag{3}$$

where angular brackets represent averages taken with respect to the Gibbs canonical ensemble, since the plasma is in equilibrium.

$$\vec{R}_j(t)=\vec{R}_j(0)+(e_j/m_j)\int_0^t d\tau\int_0^\tau d\xi\,\vec{E}_j(\xi)\cdot\hat{b}\hat{b}+c\int_0^t d\tau\,\vec{E}_j(\tau)\times\vec{B}/B^2+V_jt\hat{b} \tag{4}$$

$$\equiv\vec{R}_j(0)+\Delta R_{j\parallel}(t)\hat{b}+\Delta\vec{R}_{j\perp}(t), \tag{5}$$

say, where $\vec{E}_j(\tau)=\vec{E}(\vec{R}_j(t),t)$. The position of the test ion, initially at the origin, is given by

$$\vec{r}(t)=(e/m_i)\int_0^t d\tau\int_0^\tau d\xi\,\vec{E}(\xi)\cdot\hat{b}\hat{b}+c\int_0^t d\tau\vec{E}(\tau)\times\vec{B}/B^2+v_\parallel t\hat{b} \tag{6}$$

$$\equiv\Delta r_\parallel(t)\hat{b}+\Delta\vec{r}_\perp(t), \tag{7}$$

where m_i is the mass of the test ion of charge $+e$. V_j and $v_\parallel$ are the initial velocities along $\vec{B}$, and are distributed according to the Maxwell-Boltzmann distribution.

The ensemble average in Eq. (3) is too difficult to evaluate, and to proceed further assumptions have to be made. We first factorize the exponential of the initial locations from the perpendicular and parallel motion. The delocalization of the remaining term occurs through two processes: (1) guiding-center drift across the field lines and (2) the motion, due to the fluctuating electric fields, along the field lines. For large $\vec{B}$, $k_\parallel^2 \Delta r_\parallel^2 \gg (\vec{k}_\perp \cdot \Delta\vec{r}_\perp)^2$ on the average, for $k_\parallel \neq 0$; and similarly for the jth particle. Thus we decouple the perpendicular and parallel motions. Hence[1,5,6]

$$\langle \exp\{i\vec{k}\cdot[\vec{r}(t)+\vec{R}_i(0)-\vec{R}_j(t)]\}\rangle \cong \langle \exp\{i\vec{k}\cdot[\vec{R}_i(0)-\vec{R}_j(0)]\}\rangle \times \langle \exp(ik_\parallel \Delta r_\parallel)\rangle \langle \exp(ik_\parallel \Delta R_{j\parallel})\rangle \langle \exp(i\vec{k}_\perp\cdot\Delta\vec{r}_\perp)\rangle \langle i\vec{k}_\perp\cdot\Delta\vec{R}_{j\perp}\rangle). \tag{8}$$

The first factor in Eq. (8) can be immediately evaluated since the $\vec{R}_i(0)$ are statistically distributed according to the Gibbs ensemble. Since the electric field is assumed to be jointly normal (i.e., it is a Gaussian distribution for successive time instants), then, with the employment of the cumulant expansion[7,8] on the remaining factors of Eq. (8), it can be shown[6] that on summing the diagonal components of Eq. (3) we have

$$\frac{c^2}{B^2}Q_\perp(t) \equiv \frac{d^2R_\perp(t)}{dt^2} = \frac{\epsilon}{V}\sum_{\vec{k},k_\parallel=0}\frac{\exp[-2k_\perp^2 R_\perp(t)]}{1+k_\perp^2\lambda_D^2} + \frac{\epsilon}{V}\sum_{\vec{k},k_\parallel\neq 0} f(\vec{k},t), \tag{9}$$

$$Q_\perp(t) \equiv \tfrac{1}{2}\langle \vec{E}_\perp(0)\cdot\vec{E}_\perp(t)\rangle, \tag{10}$$

and $R_\perp(t)$, with $R_\perp(0) = dR_\perp(0)/dt = 0$, gives the mean dispersion of the test ions. Note that

$$dR_\perp(\infty)/dt = \tfrac{1}{2}D_\perp, \tag{11}$$

$$f(\vec{k},t) = \frac{1}{2}\frac{k_\perp^2}{k^2}\frac{K_D^2}{k^2+K_D^2}\exp(-k_\parallel^2 V_{\rm ion}^2 t^2 - k_\parallel^2 D_\parallel t^3), \tag{12}$$

since $m_i \gg m_e$. Here, $\lambda_D^2 = (\text{Debye length})^2 = K_D^{-2} = k_B T/8\pi n_0 e^2$ with $n_0 = N/V$, and $\epsilon = 16\pi^2 n_0 e^2 c^2/B^2 K_D^2$. $V_{\rm ion}$ is the ion thermal speed $(k_B T/m_i)^{1/2}$. The velocity diffusion coefficient $D_\parallel = (e^2/m_i^2)\int_0^\infty dt \langle E_\parallel(0)E_\parallel(t)\rangle$ can be calculated using the superposition of dressed free-streaming test particles, and we find that

$$D_\parallel = \tfrac{1}{8}\pi^{-3/2}\omega_{pi}^3\lambda_D^2\epsilon_{\rm pl}\ln\epsilon_{\rm pl}^{-1}, \tag{13}$$

where $\epsilon_{\rm pl} = 1/n_0\lambda_D^3$ is the plasma parameter, $\omega_{pi}^2 = 4\pi n_0 e^2/m_i$, and $\Omega_i = eB/m_i c$. One now sees that for $k_\parallel > K_D\epsilon_{\rm pl}\ln\epsilon_{\rm pl}^{-1}$ the free-streaming part of Eq. (12) will dominate the velocity diffusion part and vice versa for $k_\parallel < K_D\epsilon_{\rm pl}\ln\epsilon_{\rm pl}^{-1}$. Except for a short initial time,[4] $R_\perp \cong \tfrac{1}{2}D_\perp t$, and one can easily set up an iterative scheme to correct this lowest-order approximation.[9] Hence, on integrating Eq. (9), using Eq. (11), the transverse spatial diffusion coefficient $D_\perp$ is given by [on going from discrete to continuous k and allowing a cutoff $k_\perp(\min) = 2\pi/L$]

$$D_\perp = \alpha + (\alpha^2 + 2\beta)^{1/2}, \tag{14}$$

where

$$\beta = \left[\frac{1}{2}\frac{ck_B T}{eB}(2\pi n_0\lambda_D^3)^{-1/2}\left(\frac{\lambda_D}{L}\ln\frac{L}{2\pi\lambda_D}\right)^{1/2}\right]^2, \tag{15}$$

$$\alpha = \frac{1}{32\pi^2(2\pi)^{1/2}}\frac{ck_B T}{eB}\frac{\omega_{pi}}{\Omega_i}\epsilon_{\rm pl}\left[3\left(\ln\frac{1}{\epsilon_{\rm pl}}\right)^2 + 2\ln\frac{1}{\epsilon_{\rm pl}} + 2\ln\frac{1}{\epsilon_{\rm pl}}\ln\ln\frac{1}{\epsilon_{\rm pl}}\right]. \tag{16}$$

We thus see that $D_\perp$ contains two terms, one being proportional to $1/B$ and *dependent* on the volume of the plasma (i.e., the $\beta^{1/2}$ term), and the other being proportional to $1/B^2$ and independent of the plasma volume (i.e., the α term). Note that for an infinite plasma $\beta \to 0$, so that

$$D_\perp\Big|_{V\to\infty} = \frac{1}{16\pi^2(2\pi)^{1/2}}\frac{ck_B T}{eB}\frac{\omega_{pi}}{\Omega_i}\epsilon_{\rm pl}\left[3\left(\ln\frac{1}{\epsilon_{\rm pl}}\right)^2 + 2\ln\frac{1}{\epsilon_{\rm pl}} + 2\ln\frac{1}{\epsilon_{\rm pl}}\ln\ln\frac{1}{\epsilon_{\rm pl}}\right], \tag{17}$$

which exhibits the classical $1/B^2$ dependence. In the limit of very large magnetic fields, $D_\perp|_{B\to\infty} \cong (2\beta)^{1/2}$, and the magnetic fields needed for this Bohm term to dominate the classical term are not unattainable in currently proposed thermonuclear plasmas.

The transverse equilibrium conductivity coefficient can be calculated using the Kubo[10] formalism once we have written down the Liouville equation for our 3D plasma model [see Eq. (18) with the right-hand side equal to zero]. We now apply an external spatially uniform electric field $\vec{E}=\vec{E}_0\exp(-i\omega t)$ to the thermal equilibrium plasma at $t=0$. Liouville's equation becomes

$$(\partial/\partial t+H_0)D=-e^{-i\omega t}H_1D, \tag{18}$$

where

$$H_0=\sum_i\left(c\frac{\vec{E}_i'\times\vec{B}}{B^2}\cdot\frac{\partial}{\partial\vec{x}_i'}+v_{zi}\frac{\partial}{\partial z_i}+\frac{e_i}{m_i}E_{zi}\frac{\partial}{\partial v_{zi}}\right), \tag{19}$$

$$H_1=\sum_i\left(c\frac{\vec{E}_0'\times\vec{B}}{B^2}\cdot\frac{\partial}{\partial\vec{x}_i'}+\frac{e_i}{m_i}E_{z0}\frac{\partial}{\partial v_{zi}}\right). \tag{20}$$

Here a prime denotes a 2D vector (the x,y components).

We now carry out a perturbation expansion[10] in powers of $\vec{E}_0$ (or H_1) in Eq. (18) and consider only the first term (linear response). To lowest order we see that $D^{(0)}$ is just the Gibbs distribution for an equilibrium plasma. To first order we find that [with $D^{(1)}=0$ at $t=0$]

$$D^{(1)}=-e^{-i\omega t}\int_0^t d\tau\, e^{i\omega t}e^{-\tau H_0}H_1D^{(0)}. \tag{21}$$

The ensemble average of the current density $\sum_i e_i\vec{v}_i/L^3$ can now easily be calculated using $D^{(0)}$ and $D^{(1)}$:

$$\left\langle\sum_i\frac{e_i\vec{v}_i}{L^3}\right\rangle=e^{-i\omega t}\sum_{i,j}\int_0^\infty\frac{e_ie_j}{Vk_BT}\langle\vec{v}_i(0)\vec{v}_j(\tau)\rangle\cdot\vec{E}_0e^{i\omega\tau}d\tau=\langle\vec{j}\rangle_\omega e^{-i\omega t}, \tag{22}$$

where we are now dealing with 3D vectors. Thus the conductivity tensor is given by

$$\overleftrightarrow{\sigma}(\omega)=\int_0^\infty d\tau\, e^{i\omega\tau}\sum_{i,j}\frac{e_ie_j}{Vk_DT}\langle\vec{v}_i(0)\vec{v}_j(\tau)\rangle. \tag{23}$$

For the transverse components of $\overleftrightarrow{\sigma}$, the contributions from $i=j$ terms dominate[4] those with $i\neq j$ so that $\overleftrightarrow{\sigma}$ is diagonal with

$$\sigma_\perp(\omega)=\frac{2n_0e^2}{k_BT}\frac{c^2}{B^2}\int_0^\infty dt\, e^{i\omega t}Q_\perp(t). \tag{24}$$

The dc conductivity ($\omega=0$) is then given, on using Eq. (2), by

$$\sigma_{\perp\mathrm{dc}}=(n_0e^2/k_BT)D_\perp, \tag{25}$$

where $D_\perp$ is the diffusion coefficient given by Eq. (14). Equation (25) is a "generalized Einstein relation." The ac conductivity requires a numerical solution of Eq. (24), which hopefully will be done in the near future.

The author is deeply grateful to David Montgomery for many helpful suggestions concerning the effect of free streaming of $f(\vec{k},t)$.

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*Work supported by the U. S. Atomic Energy Commission under Contract No. AEC/AT-(40-1)-2598.

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