[18]R. N. Franklin, S. M. Hamberger, G. Lampis, and G. J. Smith, in *Proceedings of the Third International Conference of Quiescent Plasmas, Elsinore, Denmark, 1971* (Jul. Gjellerups Forlag, Copenhagen, Denmark, 1971), p. 144.

[19]Notice that although the theory refers to infinite plasma, we expect that with the short wavelengths used the effect of finite column radius would be unimportant.

[20]E. Ott and C. T. Dum, Phys. Fluids 14, 959 (1971).

[21]R. N. Franklin, S. M. Hamberger, G. Lampis, and G. J. Smith, Phys. Rev. Lett. 27, 1119 (1971); R. Stenzel and A. Y. Wong, Phys. Rev. Lett. 28, 274 (1972).

[22]G. Laval, R. Pellat, and M. Perulli, Plasma Phys. 11, 579 (1969); R. N. Franklin, S. M. Hamberger, G. Lampis, and G. J. Smith, Phys. Lett. 36A, 473 (1971).

Critical Exponents for Long-Range Interactions

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Critical exponents for a d-dimensional system with an isotropic n-component order parameter and long-range attractive interactions decaying as $1/r^{d+\sigma}$ ($\sigma > 0$) are derived, using the renormalization group approach, as power series in $\epsilon = 2\sigma - d > 0$ ($\sigma \neq 2$, fixed) or $\Delta\sigma = \sigma - \frac{1}{2}d > 0$ (d fixed) and, separately, to order $1/n$ for all d and $\sigma \neq 2$. For $\epsilon < 0$ the exponents have fixed ("classical") values; when $\epsilon = \Delta\sigma = 0$ fractional powers of $\ln(\Delta T/T_c)$ appear; when $\sigma > 2$ the exponents assume their short-range values.

It has been recognized for some time that long-range attractive interactions decaying, in d dimensions, as

$$-\varphi(r) \sim J(r) \sim 1/r^{d+\sigma}, \quad \sigma > 0, \tag{1}$$

should lead to values for critical exponents differing from those appropriate to short-range interactions[1] (decaying as $e^{-r/b}$ or, see below, $\sigma > 2$); furthermore, such forces can induce critical behavior in one- or two-dimensional systems where it would otherwise be absent.[2] These surmises can be demonstrated analytically for all d in the spherical model.[3] In addition, the behavior of the one-dimensional spin-$\frac{1}{2}$ Ising model has been studied by numerical extrapolation techniques.[4] Despite the fundamental theoretical interest of the problem, however, there are no results for Heisenberg or XY models, and only one other, isolated, numerical estimate for the Ising model (for $d=2$ and $\sigma=1$).[3,5]

In this note[6] the critical exponents for general d and σ ($\neq 2$) are derived for a system with an isotropic, n-component order parameter, by using the renormalization-group approach[7] and the ϵ-expansion and $(1/n)$-expansion techniques.[8-12] To present the result we define

$$\epsilon = 2\sigma - d \text{ and } \Delta\sigma = \sigma - \tfrac{1}{2}d, \tag{2}$$

and treat d as a continuous variable.[8-10] In the "classical" regime $\epsilon, \Delta\sigma < 0$ one finds, for all n,

$$\eta = 2 - \sigma, \quad \nu = 1/\sigma, \quad \gamma = 1. \tag{3}$$

On the borderline $\epsilon = 0$, $\sigma = \frac{1}{2}d$, these expressions still apply but the correlation length and susceptibility vary as

$$\xi(T) \sim t^{-1/\sigma}(\ln t^{-1})^{n'/\sigma} \text{ and } \chi(T) \sim t^{-1}(\ln t^{-1})^{n'}, \tag{4}$$

where $n' = (n+2)/(n+8)$ and $t = (T - T_c)/T_c$ is the reduced temperature deviation from critical.

In the nonclassical region $\epsilon, \Delta\sigma > 0$ one obtains for $\sigma < 2$

$$\eta = 2 - \sigma + \epsilon^2 \Theta(n, \epsilon; (2-\sigma)/\epsilon), \tag{5}$$

where for $\sigma \to 2$ we have

$$\Theta(n, \epsilon; 0) = \tfrac{1}{2}(n+2)/(n+8)^2 + O(\epsilon), \tag{6}$$

although $\Theta(n, \epsilon; w) \to 0$ as $w \to \infty$, so that for *fixed* $\sigma < 2$ the exponent η "sticks" at the classical value, $2 - \sigma$; this has been verified to $O(\epsilon^3)$ but might well be true to all orders in ϵ. The susceptibility exponent for $\sigma < 2$ is given by

$$\frac{1}{\gamma} = 1 - \left(\frac{n+2}{n+8}\right)\frac{\epsilon}{\sigma} - \frac{(n+2)(7n+20)}{(n+8)^3}\,\mathcal{A}(\sigma)\left(\frac{\epsilon}{\sigma}\right)^2 + O(\epsilon^3), \tag{7}$$

$$\mathcal{A}(\sigma) = \sigma[\psi(1) - 2\psi(\tfrac{1}{2}\sigma) + \psi(\sigma)], \tag{8}$$

where $\psi(z)$ is the logarithmic derivative of the gamma function, $\Gamma(z)$. In the range $0 \leq \sigma \leq 2$, $\mathcal{A}(\sigma)$ is well approximated by $3 - \frac{1}{4}\sigma^2$. Expressions for other exponents follow from the standard scaling relations[1]: $\gamma = (2-\eta)\nu$, etc. For fixed d the assumption of continuous dimensionality may be avoided by converting to expansions in $\Delta\sigma$; we quote only

$$\gamma = 1 + \frac{4}{d}\left(\frac{n+2}{n+8}\right)\Delta\sigma + \frac{8(n+2)(n-4)}{d^2(n+8)^2}\left[1 + \frac{2\mathcal{a}(\frac{1}{2}d)(7n+20)}{(n-4)(n+8)}\right]\Delta\sigma^2 + \cdots. \tag{9}$$

When $n \to \infty$ all these expressions agree precisely with the exact spherical model results, as expected.[3,9,13] For large n and all $\sigma < 2$, $0 < \epsilon < \sigma$, we find $\Theta(n, \epsilon; w) = O(n^{-2})$ and

$$\frac{1}{\gamma} = 1 - \frac{\epsilon}{\sigma} - \frac{8}{n}\mathcal{F}(\sigma, \epsilon)[\mathcal{G}(\sigma, \epsilon) - \tfrac{1}{2}] + O\left(\frac{1}{n^2}\right), \tag{10}$$

with

$$\mathcal{F}(\sigma, \epsilon) = \frac{[\Gamma(\frac{1}{2}\sigma)]^2\Gamma(\sigma - \epsilon)}{\sigma\Gamma(\frac{1}{2}\epsilon)\Gamma(\sigma - \frac{1}{2}\epsilon)[\Gamma(\frac{1}{2}\sigma - \frac{1}{2}\epsilon)]^2}, \tag{11}$$

$$\mathcal{G}(\sigma, \epsilon) = \frac{\Gamma(\frac{1}{2}\sigma)\Gamma(\sigma - \epsilon)\Gamma(-\frac{1}{2}\epsilon)\Gamma(\frac{1}{2}\sigma + \frac{1}{2}\epsilon)}{\Gamma(\sigma)\Gamma(\frac{1}{2}\sigma - \epsilon)\Gamma(\frac{1}{2}\epsilon)\Gamma(\frac{1}{2}\sigma - \frac{1}{2}\epsilon)}. \tag{12}$$

On expansion in powers of ϵ these formulas confirm (7) for large n.

Lastly, for $\sigma > 2$ the short-range exponents apply for all d. To order ϵ the corresponding previous results[8-10] are reproduced formally by putting $\sigma = 2$ as might be guessed[14]; however, the nonuniformity noted in (5) destroys this continuity in σ in order ϵ^2.

A comparison of the results (3) to (9) with Nagle and Bonner's[4] numerical results for spin-$\frac{1}{2}$ linear Ising chains is shown in Fig. 1. The agreement is quite encouraging except in the changeover region ($\epsilon \simeq 0, \sigma \simeq \frac{1}{2}$). However, the estimates[4] $\tilde{\eta}$ for η probably lose validity for $\sigma \lesssim \frac{1}{2}$ and it seems likely that the logarithmic factors in (4) are disturbing the numerical analysis for γ (as can be tested by constructing examples). A similar comment applies to Joyce's $\epsilon = 0$ result[3,5] $\gamma \simeq 1.13$ for $d = 2$ with $n = 1$ and $\sigma = 2$. For the Ising model ($n = 1$) in two and three dimensions our conclusions are inconsistent with an argument of Griffiths[15] which suggests that the exponents should take their short-range values[1] whenever $\sigma > 1$. However, the argument does not seem compelling and we believe the expansions can be used up to (but not at) $\sigma = 2$ for $d = 3$ and, less accurately, when $d = 2$.

To calculate the exponents we start with the reduced Hamiltonian

$$\mathcal{H}_0/k_B T = (2\pi)^{-d}\int d^d k\, u_2(\vec{k})\vec{s}_{\vec{k}}\cdot\vec{s}_{-\vec{k}} + (2\pi)^{-3d}\int d^d k \int d^d k' \int d^d k''\, u_4(\vec{k};\vec{k}',\vec{k}'')(\vec{s}_{\vec{k}}\cdot\vec{s}_{\vec{k}'})(\vec{s}_{\vec{k}''}\cdot\vec{s}_{-\vec{k}-\vec{k}'-\vec{k}''}), \tag{13}$$

where $\vec{k}$ denotes a d-dimensional momentum variable and $\vec{s}_{\vec{k}}$ is the Fourier transform of a locally defined n-component "spin" variable $\vec{s}(\vec{x})$ for the point $\vec{x}$ in Ω; an appropriate momentum cutoff (or lattice structure) is understood. The interactions are $\hat{u}_4(\vec{k};\vec{k}';\vec{k}'') = u$ (constant) corresponding to a local $|\vec{s}(\vec{x})|^4$ term,[8,10] and, via Fourier transformation of the interactions (1), for[14] $\sigma \neq 1, 2, 3, \ldots$,

$$\hat{u}_2(k) = r + j_\sigma k^\sigma + j_2 k^2 + \cdots, \quad j_\sigma, j_2 > 0. \tag{14}$$

The parameter[8,9] r varies linearly with the temperature near T_c.

Now if (A) $\sigma > 2$, the previous renormalization-group analysis[8,10] for short-range interactions applies since only the leading k^2 term in (14) matters; specifically, higher order, including spatially anisotropic and cutoff-dependent, terms in $\hat{u}_2$ and $\hat{u}_4$ damp out under successive renormalizations and do not affect the exponent values which remain as calculated[8-10]; likewise the dimensionality d can be supposed nonintegral.

When (B) $\sigma < 2$, the renormalization-group analysis may be developed along previous lines.[7,8] Thus a reduction of the momentum cutoff by a factor $b = e^{-l}$ renormalizes the length scale so that the correlation length changes as

$$\xi \Rightarrow \xi_l = \xi/b = \xi e^{-l}. \tag{15}$$

However, the spin rescaling factor, c, must now be chosen so that the coefficient of k^σ in $\hat{u}_2(\vec{k})$ for the renormalized Hamiltonian (13) remains fixed. The exponent η is then determined[7,8,16] through the relation $c^2 = b^{2-d-\eta}$. When u is small, the leading correction to the coefficient j_σ is[8,10] $O(u^2)$ and hence to first order in u the rescaling factor is, by (13) and (14), simply $b^{d+\sigma}c^2$; equating this to unity fixes c and then yields $\eta = 2 - \sigma + O(u^2)$. In differential form the renormalization-group equations for r and u to leading order then become

$$\frac{dr}{dl} = \sigma r + (n+2)\frac{u}{j+r}, \quad \frac{du}{dl} = \epsilon u - (n+8)\frac{u^2}{(j+r)^2}, \tag{16}$$

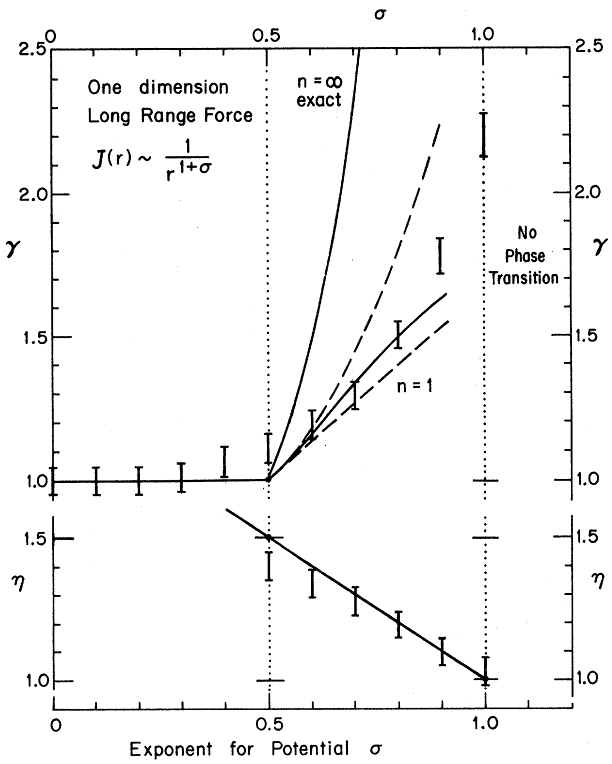

FIG. 1. Predictions for γ and η for long-range interactions in one dimension. The dashed curve and line labeled $n=1$ denote the $\Delta\sigma$ expansion (9), truncated at second order and first order, respectively; the solid curve represents the corresponding second-order ϵ expansion, for γ. The vertical bars indicate the Nagle-Bonner estimates (Ref. 4) for γ and $\tilde{\eta}$ for the spin-$\frac{1}{2}$ Ising ($n=1$) linear chain. The exponent $\tilde{\eta}$ may be identified (Ref. 4) with η for $\sigma>\frac{1}{2}$ but probably cannot be for $\sigma<\frac{1}{2}$ (where the omitted estimates rapidly level off at $\tilde{\eta}=1.5$).

where ϵ is defined as in (2), while j depends on j_σ and the momentum cutoff a^{-1} (although by choice of units we can set $j=1$ as in Refs. 7–9). To this order the momentum integrands entering the diagrammatic expansion[10] are spherically symmetric and transform trivially to short-range form by putting $k^\sigma=q^2$ and $d'=2d/\sigma$.

From (16) one first sees, as before,[7,8] that the Gaussian fixed point $r^*=u^*=0$ is stable when $\epsilon<0$. This leads directly to the classical results (3). On the borderline $\epsilon=0$ ($\sigma=\frac{1}{2}d$), this fixed point is only marginally stable and to derive (4)[17] we may, in its vicinity, neglect r in the second member of (16). The resulting solution

$$u(l)=j^2/[(n+8)(l+\bar{l})], \tag{17}$$

may be substituted into the equation for r which, on linearization, can be solved explicitly to yield an exponentially unstable part

$$r(l)\approx\bar{r}\exp\left[\sigma l-\frac{n+2}{n+8}\ln(l+\bar{l})\right], \tag{18}$$

plus transients which decay as $1/l$ for large l; the unstable amplitude, $\bar{r}$, is proportional to the temperature deviation t. These solutions need be continued only until $t_l\propto r(l)$ reaches some noncritical value of order 1, at which stage one must have $\xi_l=O(a)$. Eliminating l between (18) and (15) and using $\chi\sim\xi^{2-\eta}$ then yields (4).

When (C) $\epsilon>0$ the Gaussian fixed point is unstable (with respect to u) and a nonclassical fixed point with $u^*\approx\epsilon j^2/(n+8)$ is found. The results stated then follow from (14) to first order in ϵ by the previous arguments.[8,9] More generally, one can anticipate that if $\omega_A{}^0(\epsilon)$ is the anomalous or critical dimension[7-9] of an operator $A(x)$ for short-range interactions ($\sigma>2$), the corresponding dimension for $\sigma<2$ is $\omega_A(\epsilon)=\frac{1}{2}\sigma\omega_A{}^0(2\epsilon/\sigma)$ up to corrections of order ϵ^2 for $\epsilon>0$. To obtain the second-order terms Wilson's diagrammatic formulation[10] has been employed with k^σ replacing k^2 in the elementary propagators. It proves most straightforward to work with the order-order–energy correlation function $\hat{G}_{2s}(\vec{k},\vec{k}')$ which satisfies the matching condition[10] $\hat{G}_{2s}(\vec{k},\vec{k})/[\hat{G}(\vec{k})]^2\propto k^{(2-\eta)(\gamma-1)/\gamma}$. The calculations, while straightforward in principle, involve intricate details (the angular integrations leading, via Mellin transforms, to Legendre functions of general order) and will be published elsewhere.[18]

Finally, (D) the $1/n$ expansion has been derived following Wilson's method[11] of summing to leading order the diagrams with most closed loops. The extraction of the required logarithmic parts of the Feynman integrals, with k^σ replacing k^2 is again rather complicated.[19] (Abe's formulation[12] leads to a similar final integral.)

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[1]See, e.g., M. E. Fisher, Rep. Progr. Phys. 30, 615 (1967), Sec. 8.1. The critical exponent notation is defined in this review.

[2]The existence of a critical point in the one-dimensional Ising model for $\sigma < 1$ but its absence for $\sigma > 1$ has been proved rigorously by F. J. Dyson, Commun. Math. Phys. 12, 91, 212 (1969).

[3]G. S. Joyce, Phys. Rev. 146, 349 (1966).

[4]J. F. Nagle and J. C. Bonner, J. Phys. C: Proc. Phys. Soc., London 3, 352 (1970).

[5]C. Domb, N. W. Dalton, G. S. Joyce, and D. W. Wood, in *Proceedings of the International Conference on Magnetism, Nottingham, 1964* (The Institute of Physics and The Physical Society, London, 1965), p. 85.

[6]A summary of the results to order ϵ was presented by M. E. Fisher, at the Yeshiva University Informal Conference on Statistical Mechanics, New York, 2 May 1972 (unpublished).

[7]K. G. Wilson, Phys. Rev. B 4, 3174, 3184 (1971).

[8]K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. 28, 240 (1972).

[9]M. E. Fisher and P. Pfeuty, Phys. Rev. B 6, 1889 (1972); F. Wegner, Phys. Rev. B 6, 1891 (1972).

[10]K. G. Wilson, Phys. Rev. Lett. 28, 540 (1972).

[11]S. Ma, to be published.

[12]R. Abe, to be published.

[13]H. E. Stanley, Phys. Rev. 176, 718 (1968); M. Kac and C. J. Thompson, to be published.

[14]Note that expressions (7) to (12) do *not* apply to interactions (1) with $\sigma=2$ since this would imply $u_2(\vec{k}) = r + jk^2 \ln k + \cdots$ in place of (14) below (where the positivity of j_2 is only required for $\sigma > 2$). Similar remarks apply when $\sigma=1$ in (1). However, one may then define $J(\vec{r})$ by inversion of (14).

[15]R. B. Griffiths, Phys. Rev. Lett. 24, 1479 (1970).

[16]Compare with G. A. Baker, Jr., Phys. Rev. B 5, 2622 (1972).

[17]See also Ref. 7 and A. I. Larkin and D. E. Khmel'nitskii, Zh. Eksp. Teor. Fiz. 56, 2087 (1969) [Sov. Phys. JETP 29, 1123 (1969)].

[18]B. G. Nickel, to be published.

[19]S. Ma, to be published.

New Magnetic Phenomena in Liquid He^3 below 3 mK*

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Magnetic measurements have been made on a sample of He^3 in a Pomeranchuk cell. Below about 2.7 mK, the NMR line apparently associated with the liquid portion of the sample shifts continuously to higher frequencies during cooling. Below about 2 mK the frequency shift vanishes, and the magnitude of the liquid absorption drops abruptly to approximately $\frac{1}{2}$ its previous value. These measurements are related to the pressure phenomena reported by Osheroff, Richardson, and Lee.

Pressure measurements along the melting curve of He^3 in compressional cooling experiments have indicated the possible existence of two phase changes occurring in the He^3 within the compression cell.[1] Osheroff, Richardson, and Lee referred to these pressure phenomena as A and B. A, believed to occur at about 2.7 mK, is characterized by an abrupt decrease in the rate of cooling in the cell during a period of time in which the rate of compression is held constant. The pressure at which A occurs, $P(A)$, is highly reproducible and does not display supercooling. B, occurring at a lower temperature, perhaps 2 mK, is characterized by a sudden drop in the cell pressure by a few ten thousandths of an atmosphere upon cooling, and by a brief hesitation in the pressure as it decreases upon warming, at $P(B')$. The pressure $P(B')$ is highly reproducible, and the B' phenomenon will not occur unless the B phenomenon has already occurred. The B effect, however, shows a great deal of supercooling (as much as 10^{-2} atm), depending upon how far below $P(B')$ the cell pressure has been lowered since last going through B'. The smaller this pressure difference, the smaller will be the degree of supercooling. Although the magnetic field dependence of $P(A)$ is small and comparable in sense and magnitude to the expected depression of the melting curve itself at 2.7 mK in magnetic fields, the pressure at which B' occurs increases sharply with increasing magnetic field, and the field dependence of the pressure difference $P(B')-P(A)$ can be represented by $[P(B')-P(A)]_{H=H_0}-[P(B')-P(A)]_{H=0}=+2.02$