Damping of Finite-Amplitude Electron Plasma Waves in a Collisionless Plasma

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Experimental measurements of the collisionless damping of a finite-amplitude electron wave confirm recent self-consistent numerical solutions of the nonlinear Vlasov equation.

It has been shown theoretically[1-14] that the propagation of an electron plasma wave in collisionless plasma depends on the initial amplitude of the wave. This behavior occurs because electrons which have velocities close to the wave phase velocity (resonant electrons) become trapped in, and oscillate in, the potential wells of the wave: The number of trapped electrons and their frequency of oscillation are determined by the depths of the wells. This can lead to a decrease, or even change of sign, in the instantaneous damping rate[1-14] and to a change in the phase velocity[14] with increasing initial amplitude; or it can lead to the growth of other waves in the system,[15] the so-called sideband instability.

The theoretical approaches used to investigate the time development of an electron wave with finite initial amplitude may be broadly divided into two groups. The first method, used by Knorr,[1] Armstrong,[2] and Nührenberg,[3] is to solve numerically Vlasov's and Poisson's equations. Results show exponential damping at the linear Landau rate[16] γ_L if the initial amplitude $\varphi_0 \ll k_B T/e$ or if the observation time $t \ll \omega_B^{-1}$ $[\omega_B = k_0(e\varphi_0/m)^{1/2}$ is the initial frequency of oscillation of the electrons trapped in the wave $(\omega_0, k_0)]$; the results also show that the damping rate decreases (and may change sign, indicating wave regrowth) when φ_0 and t are increased, but for $\varphi_0 \sim k_B T/e$ even the initial damping is not exponential, the instantaneous rate exceeding γ_L.

The second approach separates the electron distribution function into resonant and nonresonant parts, and solves exactly the equations of motion for the trapped electrons. This was used by O'Neil[4] who assumed the trapped electrons to move in the potential wells of a constant-amplitude wave (i.e., $\gamma_L \to 0$), showing that after some initial Landau damping the amplitude oscillates and, because of phase mixing, reaches a constant value after a long observation time. This contrasts with the behavior implicit in the work of Al'tshul and Karpman,[5] where the amplitude oscillations persist for all times. Bailey and Denavit[6] extended the work of O'Neil to allow for a slow variation in the wave amplitude ($\gamma_L \neq 0$) for small values of the parameter $q \equiv \gamma_L/\omega_B$ and obtained results qualitatively similar to O'Neil's. Gary,[7] using a perturbation method, obtained results qualitatively similar to those of Knorr and Armstrong. All of this work, as was pointed out by Dawson and Shanny,[8] is restricted by the assumption that the slope of the initial distribution function is constant over the resonant region, and the recovery of linear Landau damping at short observation times is a consequence of this restriction. Further, none of this work is completely self-consistent as it does not include the full effect of the varying wave amplitude on the electron distribution and vice versa.

A third approach, which avoids these limitations, is by computer simulation of particle motion. Dawson and Shanny[8] find, in agreement with earlier work,[1-3] that when $\varphi_0 \sim k_B T/e$ the initial damping of the wave is not exponential, and is more rapid than that predicted by Landau.[16] Denavit and Kruer[9] show that for this condition the sideband instability occurs.

Recently, Sugihara and Kamimura,[10] extending earlier work,[11] computed self-consistent equilibrium solutions to the initial-value problem, which effectively cover the range $0 < q < \infty$ and which recover the result of O'Neil as $q \to 0$ and linear Landau damping as $q \to \infty$. (It is, however, still assumed that the slope of the initial electron distribution function in the resonant region is constant.) Their solutions extend over much longer times than in earlier treatments and show that only for $q \gtrsim 3$ does the wave damp at a constant rate γ_L; for $q > 0.77$ the damping rate decreases monotonically with time from its initial value γ_L, while for $q < 0.77$ the amplitude, after several oscillations, becomes constant with time, its actual value depending on the precise value of q. Oei and Swanson[12] have also very recently obtained self-consistent equilibrium solutions which appear to be similar to those of Sugihara and Kamimura.

The only published experimental data relevant to all this theoretical work are those by Malmberg and Wharton,[17] who observed spatial amplitude oscillations for large φ_0, in qualitative agreement with O'Neil's theory modified to the spatial situation.[13]

In this Letter we report measurements of the spatial damping of an electron plasma wave, which show in detail the transition from linear Landau damping to oscillatory behavior, and which are well described by the results of Sugihara and Kamimura[10] for the range of amplitudes below that at which sidebands appear and begin to extract significant energy from the original wave.

Our experiments were performed using a thermally ionized, collisionless, sodium plasma column, described elsewhere,[15,18] in which the electron density was $\sim 2\times10^7$ cm^{-3} with an essentially one-dimensional Maxwellian velocity distribution ($T \sim 2500$°K), and whose linear dispersion properties have been well explored.[18]

Waves were launched at $x=0$ from a short, fine-wire-probe antenna connected to the end of a matched coaxial transmission line. Resulting plasma fluctuations were detected at positions x between the transmitter and the cold end plate by a similar probe matched to the plasma with a high-input-impedance amplifier. Because of uncertainty in the probe-plasma impedance, absolute values of φ_0 could be estimated to an accuracy $\pm 20\%$ although its relative values were known more accurately ($\pm 2\%$). The spatial variation in the amplitude $\varphi(x)$ of the plasma fluctuations was recorded logarithmically using a very narrow-band ($\Delta\omega = 300$ Hz) rf receiver and an x-y recorder. Phase velocities v_φ were chosen so that there were 3 to 5 Landau damping lengths included in the length of plasma used (50 cm). This allowed the nonlinear behavior to develop sufficiently for it to be clearly distinguishable. Changes in phase velocity due to nonlinear effects[14] were undetectable ($<1\%$ for our conditions).

Figure 1 shows experimental points, taken from traces similar to those shown in Fig. 3, for the relative amplitudes of waves of the same frequency but different initial amplitudes φ_0, analyzed and plotted in terms of the dimensionless quantities used in Ref. 10, i.e., $\log(\varphi/\varphi_0)$ versus $\gamma_L t \equiv k_i x$ (for weakly damped waves $\gamma_L = k_i \partial\omega/\partial k$, where k_i is the inverse damping length[13]) with parameter q. The experimental uncertainty in k_i was $\pm 2\%$ and that in $\partial\omega/\partial k$, determined from the measured dispersion, was $\pm 5\%$, so that with the previously mentioned uncertainty in φ_0, q was known absolutely to within $\pm 17\%$. However, for a fixed frequency, the relative values of q were known to $\sim 1\%$. The results show very good agreement with the theoretically predicted curves[19] for $q \gtrsim 0.45$; in particular they demonstrate the monotonic but nonexponential decrease in amplitude for $q > 0.8$ and a transition to periodic behavior for $q < 0.6$. (The theory predicts an asymptotic stationary amplitude $\varphi^* \simeq 0.04\varphi_0$ for $q = 0.77$.) For $q \lesssim 0.04$, sidebands[15] could be detected above the background noise level; this presumably explains the greater damping suffered by the larger-amplitude waves than that predicted by the theory (which does not consider the stability of the system).

These effects can also be seen clearly in Fig. 2, which shows, for the same data as Fig. 1, the attenuation suffered by a wave in traversing a distance $x = 44$ cm ($k_i x = 4$) as a function of φ_0 compared with the theory. Only for very small

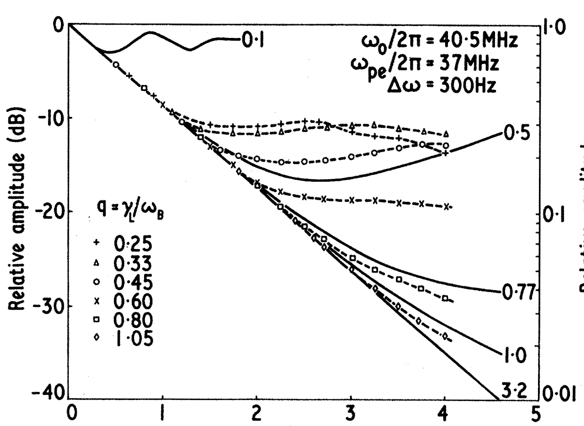

FIG. 1. Experimental data and theoretical curves (Ref. 10) showing relative spatial amplitude variations for different initial amplitudes. $k_0 = 3.64$ cm^{-1}, $k_i = 0.09$ cm^{-1}.

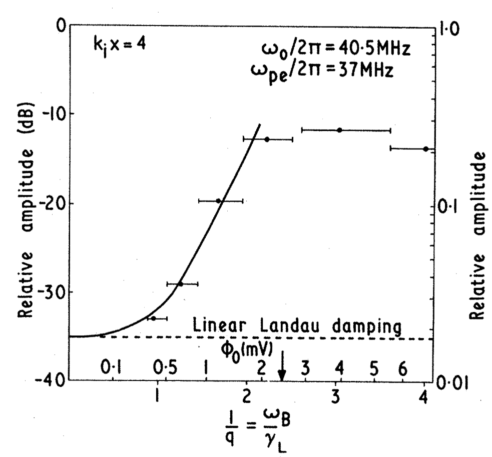

FIG. 2. Effect of different initial wave amplitudes on attenuation.

initial amplitudes ($\varphi_0 \lesssim 0.1$ mV) would the measurements agree with the linear theory, while for $\varphi_0 > 2$ mV the attenuation exceeds that of the nonlinear theory. The amplitude at which sidebands were observed is indicated with an arrow.

To demonstrate that the observed departure from linear Landau damping is caused by electrons trapped in the potential wells of the wave (ω_0, k_0), the damping of the wave was measured in the presence of a second perturbing wave (ω_1, k_1). When the amplitudes of the two waves were comparable, the only time-invariant potential well in which the electrons could be trapped traveled with a phase velocity[20] $(\omega_0-\omega_1)/(k_0-k_1)$, which was well removed from v_φ. For such conditions the amplitude variations due to trapped electrons are expected to disappear. Experimental data illustrating this are given in Fig. 3 which shows (i) a very small-amplitude wave ($e\varphi_0/k_BT \simeq 2\times10^{-3}$) exhibiting linear Landau damping, (ii) a larger-amplitude wave ($e\varphi_0/k_BT \simeq 10^{-2}$), and (iii) the same wave as in (ii) but propagating in the presence of a second perturbing wave, demonstrating that the damping is the same as that of (i).

For a one-dimensional Maxwellian distribution, the condition that the slope of the initial electron distribution function over the resonant region can be regarded as constant can be expressed as $2e\varphi_0/k_BT < [v_\varphi v_T/(2v_\varphi^2 - v_T^2)]^2$, where $v_T = (2k_BT/m)^{1/2}$. For the conditions of Figs. 1 and 2 this inequality was never seriously violated: Thus the observed initial Landau damping is to be expected. Rapid, nonexponential, initial damping[1-3,8] was not observed, and our experiments show that for most laboratory conditions damping of this sort would be obscured by other nonlinear effects, e.g., the sideband instability.[15] Furthermore, at sufficiently large amplitudes the decay of an electron wave either into a second electron wave and an ion wave[21] or into two other electron wave modes[22] would also lead to a damping rate exceeding γ_L. The spectrum was carefully checked during these measurements to ensure that none of these processes occurred.

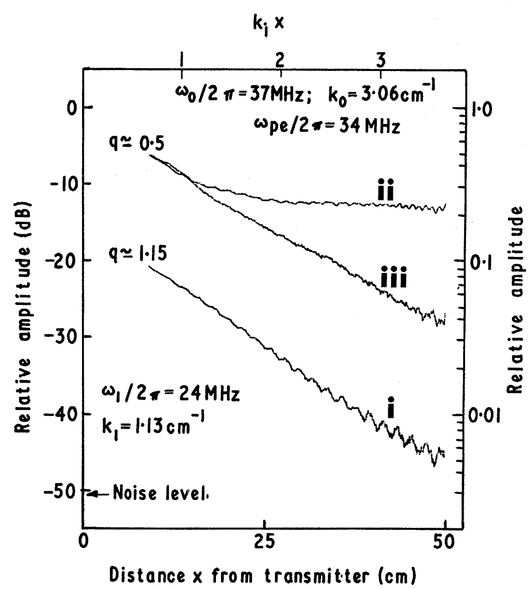

FIG. 3. Raw experimental data showing spatial amplitude variation of (i) the wave (ω_0, k_0) with $\varphi_0 \simeq 0.4$ mV; (ii) the wave (ω_0, k_0) with $\varphi_0 \simeq 2.2$ mV; and (iii) the same wave as in (ii) but in the presence of the undamped wave (ω_1, k_1) launched at $x = 67$ cm with amplitude $\simeq 4$ mV and propagating in the opposite direction to (ω_0, k_0). $k_i/k_0 = 2.4\times10^{-2}$.

Notice that for such measurements the plasma column must have a very uniform axial density. A variation of ~1% along the column would change the average linear damping length by ~20% for the conditions of Fig. 1; this effect of inhomogeneity would obscure any change in the damping caused by electron trapping.

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Critical Exponents for Long-Range Interactions

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Critical exponents for a d-dimensional system with an isotropic n-component order parameter and long-range attractive interactions decaying as $1/r^{d+\sigma}$ ($\sigma>0$) are derived, using the renormalization group approach, as power series in $\epsilon=2\sigma-d>0$ ($\sigma\neq2$, fixed) or $\Delta\sigma=\sigma-\frac{1}{2}d>0$ (d fixed) and, separately, to order $1/n$ for all d and $\sigma\neq2$. For $\epsilon<0$ the exponents have fixed ("classical") values; when $\epsilon=\Delta\sigma=0$ fractional powers of $\ln(\Delta T/T_c)$ appear; when $\sigma>2$ the exponents assume their short-range values.

It has been recognized for some time that long-range attractive interactions decaying, in d dimensions, as

$$-\varphi(r)\sim J(r)\sim 1/r^{d+\sigma},\quad \sigma>0, \tag{1}$$

should lead to values for critical exponents differing from those appropriate to short-range interactions[1] (decaying as $e^{-r/b}$ or, see below, $\sigma>2$); furthermore, such forces can induce critical behavior in one- or two-dimensional systems where it would otherwise be absent.[2] These surmises can be demonstrated analytically for all d in the spherical model.[3] In addition, the behavior of the one-dimensional spin-$\frac{1}{2}$ Ising model has been studied by numerical extrapolation techniques.[4] Despite the fundamental theoretical interest of the problem, however, there are no results for Heisenberg or XY models, and only one other, isolated, numerical estimate for the Ising model (for $d=2$ and $\sigma=1$).[3,5]

In this note[6] the critical exponents for general d and σ ($\neq2$) are derived for a system with an isotropic, n-component order parameter, by using the renormalization-group approach[7] and the ϵ-expansion and $(1/n)$-expansion techniques.[8-12] To present the result we define

$$\epsilon=2\sigma-d \text{ and } \Delta\sigma=\sigma-\tfrac{1}{2}d, \tag{2}$$

and treat d as a continuous variable.[8-10] In the "classical" regime $\epsilon,\Delta\sigma<0$ one finds, for all n,

$$\eta=2-\sigma,\quad \nu=1/\sigma,\quad \gamma=1. \tag{3}$$

On the borderline $\epsilon=0$, $\sigma=\frac{1}{2}d$, these expressions still apply but the correlation length and susceptibility vary as

$$\xi(T)\sim t^{-1/\sigma}(\ln t^{-1})^{n'/\sigma} \text{ and } \chi(T)\sim t^{-1}(\ln t^{-1})^{n'}, \tag{4}$$

where $n'=(n+2)/(n+8)$ and $t=(T-T_c)/T_c$ is the reduced temperature deviation from critical.

In the nonclassical region $\epsilon,\Delta\sigma>0$ one obtains for $\sigma<2$

$$\eta=2-\sigma+\epsilon^2\Theta(n,\epsilon;(2-\sigma)/\epsilon), \tag{5}$$

where for $\sigma\to2$ we have

$$\Theta(n,\epsilon;0)=\tfrac{1}{2}(n+2)/(n+8)^2+O(\epsilon), \tag{6}$$

although $\Theta(n,\epsilon;w)\to0$ as $w\to\infty$, so that for *fixed* $\sigma<2$ the exponent η "sticks" at the classical value, $2-\sigma$; this has been verified to $O(\epsilon^3)$ but might well be true to all orders in ϵ. The susceptibility exponent for $\sigma<2$ is given by

$$\frac{1}{\gamma}=1-\left(\frac{n+2}{n+8}\right)\frac{\epsilon}{\sigma}-\frac{(n+2)(7n+20)}{(n+8)^3}\,\mathfrak{A}(\sigma)\left(\frac{\epsilon}{\sigma}\right)^2+O(\epsilon^3), \tag{7}$$

$$\mathfrak{A}(\sigma)=\sigma[\psi(1)-2\psi(\tfrac{1}{2}\sigma)+\psi(\sigma)], \tag{8}$$

where $\psi(z)$ is the logarithmic derivative of the gamma function, $\Gamma(z)$. In the range $0\leq\sigma\leq2$, $\mathfrak{A}(\sigma)$ is well approximated by $3-\frac{1}{4}\sigma^2$. Expressions for other exponents follow from the standard scaling relations[1]: $\gamma=(2-\eta)\nu$, etc. For fixed d the assumption of continuous dimensionality may be avoided by