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[17]If one assumes that $\Phi \propto E^2 \propto r^2$ in a spherically symmetrical well about a nodal point of the field, then the mean value of Φ in a sphere of radius r_w can readily be shown to be $\frac{3}{5}\Phi_w$, where Φ_w is the quasipotential at r_w. Since the energy gain per collision can be taken to be twice this mean energy, the ratio of mean energy after a collision to that before is $\frac{11}{5}$. Thus, the number of collisions necessary for an electron to have its energy increased from an initial value u_0 to a value Φ_R is $n = [\ln(\Phi_R/u_0)]/[\ln(\frac{11}{5})]$. For $u_0 = 1$ eV and $\Phi_R = 15$ eV, $n = 3.6 \approx 4$.

Convection-Driven Hydromagnetic Dynamo

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We describe a hydromagnetic dynamo utilizing Bénard convection between rotating parallel planes. The model is based upon an asymptotic expansion in two spatial scales valid for large Taylor number.

It has long been believed that thermal convection provides a plausible thermodynamic basis for a homogeneous dynamo effect, that is, for the spontaneous amplification of magnetic energy within an isolated volume of conducting fluid.[1,2] In this Letter we consider a hydromagnetic dynamo model based upon this hypothesis. The model utilizes a magnetohydrodynamic Boussinesq fluid with constant properties, confined between stress-free, perfectly conducting, isothermal planes $z = 0, L$. The system rotates about the vertical z axis with constant angular velocity. If the magnetic field is zero, this is the classical Bénard problem with rotation. In that case it is well known that for large Taylor number $T = 4\Omega^2 L^4 \nu^{-2}$ and fixed Prandtl number $\nu\kappa^{-1} > 1$, the critical value of the Rayleigh number $R = \alpha\beta L^4(\kappa\nu)^{-1}$ is of order $T^{2/3}$, the most unstable horizontal wavelengths of the convection pattern being of order $T^{-1/6}L$.[3] Accordingly, we set $\epsilon = l/L = T^{-1/6}$, with l the horizontal scale, and suppose that $\epsilon \ll 1$, $\epsilon^4 R \equiv \tilde{R}(\epsilon) \sim 1$. We are then able to use the multiple-scale theory of spatially periodic kinematic dynamos to study the regenerative effect of the flow.[4] If convection ensues as a pattern of two or more nonparallel systems of rolls, the motion is a periodic dynamo of degree two with respect to z-dependent dynamo waves. Consequently, we require the magnetic Reynolds number of the rolls to be of order $\epsilon^{1/2}$, and take the unit of speed to be

$\eta L^{-1}\epsilon^{-1/2}$ (where η is the magnetic diffusivity). Expansions then proceed in powers of $\epsilon^{1/2}$ (or $T^{-1/12}$). We take $\eta\kappa^{-1}$ to be fixed and of order 1.

The vertical velocity component has the form (we now use dimensionless variables)

$$w=\sum_K \tilde{w}(z,t,\vec{k})\exp(i\vec{k}\cdot\vec{x}/\epsilon), \tag{1}$$

where K is a fixed symmetric set of horizontal wave-number vectors measured in units of l^{-1}. We refer to the convection field corresponding to two conjugate terms of (1) as a roll system, or simply a roll. If we set $\tilde{w}=\hat{w}(t,\vec{k})\sin\pi z$, the induction equation for the horizontal mean dynamo wave $(B_1(z,t),B_2(z,t),0)$ has the ϵ limit

$$\frac{\partial B_i}{\partial t}+2\pi\Lambda\frac{\partial}{\partial z}[\sin(2\pi z)M_{ij}B_j]-\frac{\partial^2 B_i}{\partial z^2}=0;$$

$$M_{ij}=\begin{pmatrix}-\alpha_{21} & -\alpha_{22}\\ \alpha_{11} & \alpha_{12}\end{pmatrix},\quad \Lambda=\frac{1}{4k^2}\sum_K|\hat{w}(t,\vec{k})|^2, \tag{2}$$

$$\alpha_{ij}=(2k^2\Lambda)^{-1}\sum_K(k_ik_j/k^2)|\hat{w}(t,\vec{k})|^2,$$

where time is in units of the magnetic diffusion time $\tau=L^2\eta^{-1}$. We require that $\partial B_i/\partial z$ vanish at $z=0,1$ and that the vertical mean of B_i vanish. The eigenvalues Λ corresponding to stationary solutions are then zeros of the Bessel function J_0 ($\Lambda_{\min}=2.405$). However, the minimum eigenvalue of oscillatory modes [proportional to $\exp(-i\omega t)$] is $\Lambda_c=1.5974$ with $\omega_c/\pi^2=1.3936$. Hence, regarded simply as a kinematic dynamo, the most easily excited magnetic modes are oscillatory.

We expect dynamo action on a weak seed magnetic field when the initial convection is sufficiently vigorous and irregular. Eventually, however, the magnetic stresses cannot be neglected, and it is then not clear that the requisite structure will persist indefinitely. Here, as in the nonmagnetic problem, the ultimate configuration is presumably parametrized by $(\tilde{R}-\tilde{R}_c)/\tilde{R}_c\equiv\delta(\epsilon)$, where $\tilde{R}_c=k^4+(\pi/k)^2$ is the asymptotic critical value for stationary convection at a wave number k. The Hartmann number $M(\epsilon)=B_0L(\mu\rho\nu\eta)^{-1}$ measures the intensity of the generated field, and there are three distinct nominal parameter orderings: (i) weak field, $\delta\sim\epsilon$, $M\sim1$; (ii) intermediate field, $\delta\sim\epsilon$, $M\sim\epsilon^{-1/2}$; and (iii) strong field, $\delta\sim\epsilon^2M^2\ll1$, $\epsilon M^2\gg1$. The ratio of magnetic to kinetic energy density in these three cases is of order ϵ, 1, and ϵM^2, respectively. In the weak-field limit the flow is geostrophic at the lowest (order 1) level, and is determined recursively by solving an inhomogeneous form of the classical Bénard equations. The weak-field equations given below are then obtained as a solvability condition on the Bénard problem at the ϵ^3 level. The first nonlinear effect is of order ϵ^2, and is due to the reduced bouyancy associated with the first perturbation of the mean temperature; this perturbation is measured in units βL and has the form $\Theta(z,t)=\hat{\Theta}(t)\sin 2\pi z$. Setting $\delta/\epsilon=\hat{\delta}=\delta^{(0)}+\epsilon M^2\delta^{(1)}+\cdots$ and $\hat{\Theta}/\epsilon=\hat{\Theta}^{(0)}+\epsilon\hat{\Theta}^{(1)}+\cdots$, there results $\delta^{(0)}=-\pi\hat{\Theta}^{(0)}=(\eta/\kappa)^2\Lambda=\text{const}$. For rolls of fixed wave number k the kinetic energy of the convection is proportional to Λ and is therefore fixed by $\delta^{(0)}$. (This can also be interpreted as a constraint on the convective transport of heat between the two isothermal planes, the Nusselt number here being $N=1-2\pi\epsilon\hat{\Theta}^{(0)}+\cdots$.) On this energy surface the functions $\hat{w}(t,\vec{k})$ are determined (with k now fixed) by

$$d\hat{w}(t,\vec{k})/dt+\{|w(t,\vec{k})|^2\sum_{K'}[A(\vec{k}',\vec{k})-cC(\vec{k}',\vec{k})]\}\hat{w}(t,\vec{k})=0, \tag{3}$$

where A and C are skew in their arguments, and c is a positive constant. The coupling with B_i is through C:

$$C=k^{-2}(k_ik_j-k_i'k_j')m_{ij},$$
$$m_{ij}=(\pi^2+k^6)^{-1}\int_0^1(\pi^2\cos^2\pi z-k^6\sin^2\pi z)B_iB_j\,dz. \tag{4}$$

Equations (1)–(4) together with the energy constraint determine B_i and w in the weak-field limit. The number of terms permitted in (1) is of course arbitrary, but rolls in at least two directions are needed to make M in (2) of rank 2. The disappearance of all rolls but one leads to decay of B_i and the dynamo fails. A promising example is discussed below.

The intermediate-field limit gives equations connecting the $\hat{w}(t,\vec{k})$, $\hat{\Theta}^{(0)}(t)$, and $\Lambda(t)$ at the ϵ^2 level:

$$[\hat{\delta}+m_{ij}k_ik_j+\pi\hat{\Theta}^{(0)}]\hat{w}(t,\vec{k})=0, \tag{5a}$$

$$\partial\hat{\Theta}^{(0)}/\partial t+4\pi\eta\kappa^{-1}\Lambda+4\pi^2\kappa\eta^{-1}\hat{\Theta}^{(0)}=0. \tag{5b}$$

Inertial response of velocity can occur here only on a fast time scale $\epsilon\tau$, but Λ now evolves on the diffusion scale τ. The strong-field limit reproduces (5a) without $\pi\hat{\Theta}^{(0)}$, and (5b) is disregarded.

The ϵ limit of stationary equilibria in the strong-field regime can be studied for ϵM small but finite. It follows from (5a) that for given B_i the tions can be satisfied for no more than three rolls in general (unless $\hat{\delta}$ is assigned a dependence upon $\vec{k}/k$). Vanishing of one bracketed factor in the equations can be satisfied for no more than three rolls in general (unless $\hat{\delta}$ is assigned a dependence upon $\vec{k}/k$). Vanishing of one bracketed factor in the equations (5a) gives one constraint on B_i, and the condition that this constraint be satisfied for all time determines in principle one function of time in w. The situation in this case seems therefore to be analogous to Taylor's condition for spherical dynamos.[5] For the special case of two orthogonal rolls, a straightforward bifurcation analysis establishes the existence of stationary equilibria, provided that Λ is near a root of J_0 and that the mean of B_i vanishes to order (ϵM).[4] If the latter condition is not satisfied, it can be shown by formal perturbation that nonstationary solutions exist, oscillating on the slow time $\tau(\epsilon M)^{-2}$. These solutions, stationary and nonstationary, are, however, believed to be overstable on the diffusion time because of the value of Λ_c noted previously.

The evolution of a weak field has not yet been studied in detail, and it is not known whether or not an intermediate field will persist as an oscillation having period $O(\tau)$. To study two orthogonal rolls in the weak-field limit, we set $|\hat{w}(t, \vec{k}_{1,2})|^2 = 2(k\kappa/\eta)^2\delta^{(0)}(1 \pm 2v)$ in order to satisfy the energy constraint, in which case (3) reduces to the single equation

$$dv/dt + [2A(\vec{k}_1, \vec{k}_1) + c(m_{22} - m_{11})](\tfrac{1}{4} - v^2) = 0. \quad (6)$$

We note from (6) that $v \to \pm\frac{1}{2}$ if $m_{11} = m_{22} = 0$; i.e., without the magnetic field the convection collapses onto a single roll. We suppose that v is near zero (of order v_0), and that $\Lambda - \Lambda_c \sim v_0$, $\varphi \equiv (\frac{1}{2}+v)^{1/2}B_1 + i(\frac{1}{2}+v)^{1/2}B_2 \sim v_0^{1/2}$. We also neglect $2A$ in (6). Indeed, this term is indentically zero for a suitable choice of rolls with slightly different wave numbers. However, the main justification for this simplification in a preliminary calculation is that the dynamical effect of the magnetic field is thereby isolated. We then find that, to lowest order, $\varphi = v_0^{1/2} f(t^*)\varphi_c(z)\exp(-i\omega_c t)$, where $t^* = v_0 t$. Since v is periodic of period ω_c/π to lowest order, there is an effect on induction in (2) of order $v_0^{3/2}$. The terms of this order in (2) then give an equation for $x(t^*) \equiv |f|^2$ of the form

$$dx/dt^* = (ax - b)x. \quad (7)$$

Here the constant a is positive when $k = k_c = (\frac{1}{2}\pi^2)^{1/6}$. The sign of b (a constant representing the effect of v) is that of $\Lambda_c - \Lambda$. If $b > 0$, we see that $x \to 0$ as $t^* \to \infty$ if x is initially in the interval $0 < x < b/a$, and $x \to \infty$ as $t^* \to \infty$ if x is initially greater than b/a or if $\Lambda > \Lambda_c$. In the former case the dynamo fails. In the latter case all that can be said is that the field initially grows and persists for many diffusion times. We are unable to predict in this example the behavior of the motion once the magnetic field has grown to its nominal level and $M \sim 1$. If the weak-field limit admits magnetic modes which grow indefinitely, the intermediate-field limit would presumably describe the system once magnetic stresses dominate the effects of advection, with fluctuations of the kinetic energy occurring on a diffusion time.

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