"effective pole" does not come in by naively replacing the cut by a pole. It results from the threshold behavior deduced from the unitarity relations plus physical assumptions.

As an illustration, we give a numerical example of how the forward peak may look different from an exponential form. Since we have only deduced the polelike threshold behavior for $D(t)$, and for large t we only know that $D(t)$ approaches a constant, we simply take as the example

$$D_R^{NN}(t) \propto [1/(5m_\pi^2 - t) + \mathrm{const}]. \qquad (13)$$

Taking $f_0(t) \propto e^{4t}$ as empirically observed at larger t, we find the result as shown in Fig. 1, which is consistent with the experimental data.[7]

We remark that the same threshold behavior should also appear in πN data according to our argument. If the threshold behavior deduced here were indeed to be important, it would unfortunately also make it more difficult to determine the total cross section precisely by measuring the nuclear-Coulomb interference, because the polelike appearance of the threshold behavior resembles that of the Coulomb amplitude.

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Threshold Enhancements and the Statistical Bootstrap of Hadrons*

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A method of calculating inelastic hadronic processes is proposed in which massive states are produced by Regge exchange and decay into final states with coupling strengths given by the statistical bootstrap of Frautschi. The invariant-mass distributions for a particular final state are peaked slightly above threshold and fall off exponentially with increasing mass. Good agreement with experiment is obtained for numerous reactions.

Two recent theoretical developments are utilized in this paper to provide a description of hadronic inelastic processes. One development that is adopted here is that massive states are produced by the exchange of the Regge singularities.[1,2] Simple forms for the decay amplitude of the massive states into low-mass objects have led to an adequate description of inclusive spectra. We assume further that the hadrons produced in this manner are composite states of hadrons generated by the statistical bootstrap of Hagedorn[3] and Frautschi.[4] This is the other recent development invoked, and this assumption leads naturally to a statistical description of the decay matrix of the high-mass state into any specified final state. One then has a framework for calculating exclusive as well as inclusive processes.

Using this procedure, we have calculated mass distributions for a large number of states (results will be presented below). We find good agreement with measured spectra even though the mass region considered is not very high. The spectral shape of the A regions of the $\rho\pi$ and $f\pi$ spectra, the Q region of the $K^*(890)\pi$ spectrum, and the L region of the $K^*(1420)\pi$ spectrum are explained by this theory. The shapes of the $n\pi$ and Δn spectra are described too.

We also look at the $\pi\pi$ spectrum. At the low energies encountered in the $\pi\pi$ spectrum one would not expect the density of states to be great enough for the statistical model to give a good description of the data. The theoretical spectral shape does give a crude averaged picture with the ρ peak appearing as a large fluctuation.

In the following, the statistical assumption is introduced for the decay matrix, and a formula is written down for the mass distribution for the decay of a massive state produced by Regge exchange into a two-body final state. Experimental data for several reactions are compared with the model using the density of states of the statistical bootstrap. No adjustment of the parameters determined by Frautschi and Hamer[5] is necessary in order to fit the data. An arbitrary scale factor is used to normalize to each set of measurements. Though constraints exist between the scale factors in principle, we do not consider this question here. The effects of spin have been ignored, and we made no attempt to fit angular distributions. The extension of the statistical bootstrap to determine the density of states as a function of angular momentum has been discussed to some extent[6] and is necessary should one wish to describe the angular distributions of the produced particles. We do not pursue this point here.

Consider production by Regge exchange in which one of the incident particles is excited to a high-mass state which decays into a final state labeled by f. The distribution in m, the total invariant mass of the final state, is given by

$$\frac{d\sigma_f}{dm^2}=\frac{\pi}{4(2\pi)^5\lambda(s,m_a^2,m_b^2)}\int dt\,R(s,t,m)\int\omega_f(P,p_1,\ldots,p_{N_f})\delta^4(P-\sum_i p_i)(2\pi)^{-3N_f}\prod_i\frac{d^3p_i}{2E_i}. \tag{1}$$

Here P is the four-momentum of the massive state, and N_f is the number of particles in the state f. The p_i are the momenta of the decay products and $R(s,t,m)$ is the modulus of the Regge exchange amplitude squared, s and t being the usual Mandelstam variables. ω_f is the square modulus of the decay amplitude of the state with mass m into the final state f. We take $\lambda(x,y,z) = x^2+y^2+z^2-2xy-2xz-2yz$.

We now restrict ourselves to cases where m decays into a two-body final state. According to the statistical bootstrap, such decays are predominant over higher-multiplicity states, pion plus heavy resonance emission being the most common in general. It should be emphasized, however, that resonances and particles are treated on an equal footing. Hence, the produced particle can decay into unstable particles which undergo secondary decays.

To determine ω_f, we assume that the decay rates into different final states are given by the densities of these states within the hadron of mass m in the statistical bootstrap theory, divided by the total density of states at that mass. Then, for the decay into specific resonances of masses m_1 and m_2,

$$\omega_f(P,p_1,p_2)=A\Gamma(m,m_1,m_2)/\rho(m), \tag{2}$$

where A is a constant and

$$\Gamma(m,m_1,m_2)=[m^4-(m_1^2-m_2^2)^2/4m^2]. \tag{2a}$$

Γ is the ratio of noninvariant two-body phase space to invariant phase space evaluated in the rest frame of the massive state of four-momentum P. Since the statistical bootstrap is formulated in terms of noninvariant phase space and we have used invariant phase space in formulating Eq. (1), a factor of Γ is needed to compensate.

The assumption that the transition matrix is inversely proportional to the density of states at mass m is a familiar one in nuclear physics. In fact, a form similar to (2) describes neutron capture γ-ray spectra.[7] In the theory presented here, if one sums over all possible final states, the numerators sum to $\rho(m)$, so that $d\sigma/dm$ is independent of the density of states. This fact is consistent with the observation that missing-mass spectra do not show the exponential falloff characteristic of $\rho^{-1}(m)$.

In general, ω_f will depend on the relative angular coordinates of the secondaries and the spin J of the state produced. In the present paper, we are not interested in producing detailed fits to the angular distributions. Rather, our purpose is to demonstrate how a statistical theory of decay, based on the density of states from the statistical bootstrap, will explain the coupling strengths of the various channels as a function of the mass of the state produced. Since the distributions including spin are proportional to the distribution used here,[6] when one integrates over angular coordinates the decay matrix used here would appear, multiplied by an additional factor which is weakly dependent on m.[6]

Evaluating the integral over phase space gives

$$\frac{d\sigma_f}{dm} = \frac{A\Gamma(m, m_1, m_2)}{\rho(m)} \frac{[\lambda(m^2, m_1^2, m_2^2)]^{1/2}}{m} \left\{ \frac{\pi}{16(2\pi)^{11}\lambda(s, m_a^2, m_b^2)} \int dt\, R(s, t, m) \right\}. \quad (3)$$

The factor in braces involves the Regge scale factor and residue function at the upper vertex, which may be evaluated for 1 $\mathrm{GeV}^2 \ll m^2 \ll s$ using Mueller's theorem.[8] We will use this formula at low masses, and we assume it to be a slowly varying function of m. As $s \to \infty$, the factor in brackets will approach a constant (to within logarithms) if Pomeranchukon exchange is allowed. Otherwise it will fall as a power of s.

We will analyze reaction data in experiments of the type

$$a + b \to c + m, \quad m \to \pi + d, \quad (4)$$

where d is a stable particle or resonance. According to formula (3)

$$\frac{d\sigma_f}{dm} \simeq \text{const} \frac{\Gamma(m, m_\pi, m_d)[\lambda(m^2, m_\pi^2, m_d^2)]^{1/2}}{m\rho(m)}, \quad (5)$$

and we will use this form with $\rho(m) = (\text{const}/m^3) \times \exp(m/m_\pi)$, following Frautschi's statistical bootstrap result.[4,5] The specific data correspond to the reactions

$\pi + p \to (\rho\pi) + p \to 3\pi + p$ (A_1, A_2 region),

$\pi + p \to (f\pi) + p \to 3\pi + p$ (A_3 region),

$K + p \to [K^*(890)\pi] + p \to K + 2\pi + p$ (Q region),

$K + p \to [K^*(1420)\pi] + p \to K + 2\pi + p$ (L region),

$\pi + p \to \pi + (n\pi) \to 2\pi + n$,

$\pi + p \to \pi + (\Delta\pi) \to 3\pi + p$,

$\pi + p \to (\pi\pi) + \Delta \to 3\pi + p$ (ρ, f_0 region).

The calculations and results are shown in Figs. 1 and 2. All but the reaction $\pi + p \to (\pi\pi) + \Delta$ have a threshold above 1 GeV, and the spectrum of resonances, it would appear, is rich enough to make for validity of the statistical approach used here. The agreement with experiment is good. For the case of $\pi + p \to (\pi\pi) + \Delta$ the statistical model seems to produce a crude average behavior. The prominent ρ resonance appears as a strong fluctuation about this average and the f^0 as a lesser fluctuation. This latter reaction is the only one of those considered above that cannot go by Pomeranchukon exchange, and we do not know whether or not it is so that Regge exchange is more selective than Pomeranchukon exchange in exciting resonances. At any rate, the threshold for this reaction is low and we do not expect statistical considerations to be valid.

We have provided a procedure for calculating production processes which is clearly capable of describing the results of a wide class of strong interactions. This picture provides an alternative to the Deck mechanism and is extremely simple.

It would appear from the above that the A_1, A_3,

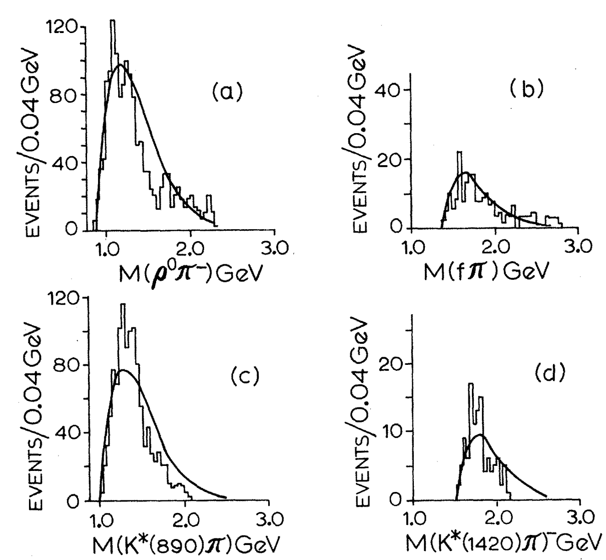

FIG. 1. Predictions of the statistical model for (a) the $\rho\pi$ mass distribution in the A_1, A_2 region of the $\rho\pi$ spectrum compared with the 16-GeV/c $\pi^- + p \to (\rho\pi)^- + p$ data of Ref. 9; (b) the $f\pi$ mass distribution in the A_3 region of the $f\pi$ spectrum compared with the 16-GeV/c $\pi^+ + p \to (f^0\pi) + p$ data of Ref. 9; (c) the $K^*(890)\pi$ mass distribution in the Q region of the $K^*(890)\pi$ spectrum compared with the 4.6-GeV/c $K^- + p \to [K^*(890)\pi]^- + p$ data of Ref. 10; (d) the $K^*(1420)\pi$ mass distribution in the L region of the $K^*(1420)\pi$ spectrum compared with the 4.5-GeV/c $K^- + p \to [K^*(1420)\pi]^- + p$ data of Ref. 10.

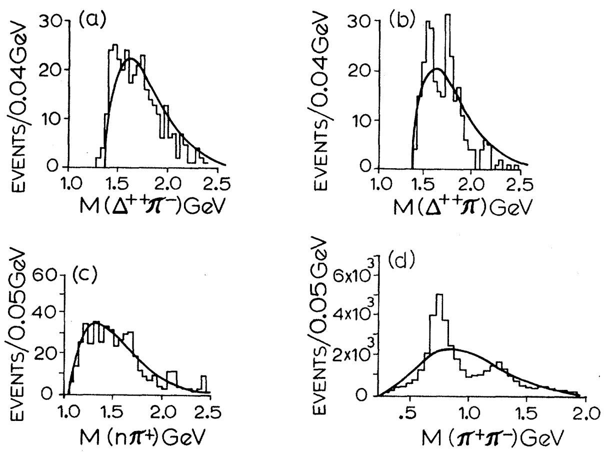

FIG. 2. Predictions of the statistical model for (a) the $\Delta^{++}\pi$ mass distribution compared with the 16-GeV/c $\pi^- + p \rightarrow \pi^- + (\Delta\pi)^+$ data of Ref. 9; (b) the $(\Delta\pi)^+$ mass distribution compared with the 16-GeV/c $\pi^+ + p \rightarrow \pi^+ + (\Delta\pi)^+$ data of Ref. 9; (c) the $n\pi^+$ mass distribution compared with the 16-GeV $\pi^- + p \rightarrow \pi^- + (n\pi^+)$ data of Ref. 9. (d) the $\pi^+\pi^-$ mass distribution in the ρ-f region of the $\pi\pi$ spectrum compared with the 7-GeV/c $\pi^+ + p \rightarrow (\pi^+\pi^-) + \Delta^{++}$ data of Ref. 11.

Q, L, etc., bumps could be the cumulative effect of a density of resonances which overlap. This is, so far as we know, a new point of view, and since these resonances are likely to have sizable widths, it would be difficult in general to see their components as discrete states in strong interactions. We point out at this time that since we have not separated angular momentum states in our calculations, it is reasonable that there could be several or many states in the regions of m considered. It is to be noted for example that analysis in the A_1, A_2 region yields substantial components for several values of angular momentum.[12]

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