and of course zero asymmetry if both foil and wire were heated to the same temperature. The results shown in Figs. 2(a) and 2(b) illustrate these results, although there is a small additional asymmetry which is independent of temperature gradient.

Successful measurements of the asymmetry were frequently frustrated by the mechanical and electrical instability of the point-contact junctions. Small mechanical vibrations or electrical switching transients caused discontinuous and irreversible changes in the I-V characteristics of the junction. However, seventeen different asymmetry measurements were made. The magnitude of the measured asymmetry varied from 8 times less than that predicted by Eq. (3) to almost 7 times greater. This spread can be attributed to the widely different structures obtained each time the junction was remade. Thus, the effective length and area of the junction, the amount of oxide between the two surfaces, and the straining of the metal were all likely to vary enormously. In particular, the region across which the temperature gradient existed was probably very ill-defined, so that the effective quasiparticle resistance R was also correspondingly badly defined, and not necessarily given by the resistance of the I-V characteristic. With these factors in mind, we feel that the spread in our data is acceptable. It is important to note that although the magnitude of the asymmetry was very irreproducible, the sense was always as expected, a fact which gives us confidence that we were indeed observing the superfluid backflow.

In summary, we have observed an asymmetry in the critical current of a Pb-Pb point contact induced by a temperature difference across the junction. We ascribe this asymmetry to the counterflow of supercurrent analogous to the counterflow of superfluid helium observed in the fountain effect. The asymmetry was always in the expected direction, and of approximately the expected magnitude.

We should like to thank Professor Richard E. Packard for a useful discussion and a critical reading of the manuscript.

This work was supported by the U. S. Atomic Energy Commission.

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Feynman-Graph Expansion for the Equation of State near the Critical Point (Ising-like Case)*

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(Received 16 June 1972)

The scaling equation of state of an Ising-like ferromagnet is derived by an expansion in $\epsilon = 4-d$, where d is the dimension of space. The result is compared with numerical calculations on the three-dimensional Ising model. It is also established that the "linear model" is exact up to order ϵ^2.

The Feynman-graph method used previously for calculating critical exponents[1] is used here to obtain the equation of state near the critical point. It is calculated in an expansion in $\epsilon = 4-d$, where d is the dimension of space. The calculation is performed for an Ising-like ferromagnet, but the result may be applied to liquid-gas transitions and other critical points by relabeling the variables.[2] The equation of state is obtained in the scaling form predicted by Widom and others. In terms of the magnetic field H, the magnetization M, and the reduced temperature $t = (T - T_c)/T_c$, a scaling form is[2,3]

$$H/M^{\delta}=f(t/M^{1/\beta}). \tag{1}$$

In this work, the critical exponents δ and β and an explicit expression for the function f have been obtained to order ϵ^2.

The exponents have been derived for an arbitrary number n of spin components as in Ref. 1:

$$\beta=\tfrac{1}{2}-\frac{3}{2(n+8)}\epsilon+\frac{(2n+1)(n+2)}{2(n+8)^3}\epsilon^2+O(\epsilon^3), \tag{2}$$

$$\delta=3+\epsilon+\frac{n^2+14n+60}{2(n+8)^2}\epsilon^2+O(\epsilon^3). \tag{3}$$

These results agree with what one obtains from the scaling relations[2]

$$\beta=\gamma(d-2+\eta)/2(2-\eta), \tag{4}$$

$$\delta=1+2(2-\eta)/(d-2+\eta), \tag{5}$$

and the values of γ and η previously calculated in Ref. 1. In three dimensions and for $n=1$, this gives the values $\beta=0.34$, $\delta=4.46$.

The function f is reported only for the Ising-like case $n=1$. For $n>1$, the appearance of a massless Goldstone mode below T_c leads to infrared problems which require a separate analysis.[4]

Since the method of calculation has been described in a previous work,[1] we will only review briefly the notations and the basic steps. The Hamiltonian used is written

$$\mathcal{H}/kT=\int d^dx\,[\tfrac{1}{2}\nabla^2 s(x)+\tfrac{1}{2}r_0s^2(x)+(u_0/4!)s^4(x)-Hs(x)]. \tag{6}$$

It is convenient to subtract from $s(x)$ its expectation value, and to define

$$\varphi(x)=s(x)-M, \tag{7}$$

$$M=\langle s(x)\rangle, \tag{8}$$

where the bracket denotes the thermodynamic average. In terms of the field φ, the Hamiltonian is then split into two pieces, $\mathcal{H}=\mathcal{H}_0+\mathcal{H}_1$, with

$$\begin{aligned}\mathcal{H}_0/kT&=\tfrac{1}{2}\int d^dx\,(\nabla\varphi^2+r\varphi^2),\\ \mathcal{H}_1/kT&=\int d^dx\,[(u_0/4!)\varphi^4+\tfrac{1}{6}u_0M\varphi^3+\tfrac{1}{2}(r_0-r+\tfrac{1}{2}u_0M^2)\varphi^2+(r_0M+\tfrac{1}{6}u_0M^3-H)\varphi],\end{aligned} \tag{9}$$

where r is the reciprocal of the susceptibility, and $\mathcal{H}_1$ is treated as a perturbation.

Instead of using a cutoff equal to 1 as in Ref. 1, all propagators have been cut off at momentum Λ, much larger than the inverse of the correlation length, in order to check that all physical quantities are indeed independent of Λ.

The first step is to determine the coupling constant u_0,[1]

$$u_0=\tfrac{16}{3}\pi^2\epsilon+\tfrac{16}{3}\pi^2\epsilon^2(A+\tfrac{1}{2}\ln\Lambda^2)+O(\epsilon^3), \tag{10}$$

$$A=\tfrac{17}{54}-\tfrac{1}{2}\ln 4\pi+\tfrac{1}{2}c\simeq-0.847 \tag{11}$$

(c denotes Euler's constant). The explicit cutoff dependence of u_0 is not unexpected. At order ϵ, u_0 does coincide with the fixed-point solution of the renormalization group equations[1,5] but not at order ϵ^2.

The spontaneous magnetization below T_c turns out to be of order $r^{1/2}\epsilon^{-1/2}$ and, therefore, to do a consistent calculation, factors of u_0M^2/r must be regarded as of order 1.

Expanding the relation $\langle\varphi\rangle=0$ in powers of ϵ, we obtain

$$\frac{H}{M}=t+\frac{u_0M^2}{6}+\frac{u_0}{32\pi^2}r(\ln r-\tfrac{1}{4}\epsilon\ln^2 r-\ln\Lambda^2)+\frac{u_0{}^3M^2}{2^{10}\pi^4}[\ln r(\ln\Lambda^2-2)-\tfrac{1}{2}\ln^2 r]-\frac{u_0{}^2r}{2^{10}\pi^4}[2\ln r(\ln\Lambda^2+1)-\ln^2 r]. \tag{12}$$

This equation and the relation

$$r=[\partial H/\partial M]_t \tag{13}$$

provide an implicit definition of the equation of state. Defining

$$x=t/M^{1/\beta}, \tag{14}$$

it is tedious but simple to show that all $\ln M$ and $\ln\Lambda^2$ dependence disappears, provided a suitable renormalization of the temperature scale is performed. The result reads

$$H/M^{\delta}=x+\tfrac{1}{6}u+\tfrac{1}{6}\epsilon[(x+\tfrac{1}{2}u)\ln(x+\tfrac{1}{2}u)+Au]+\epsilon^2\{\tfrac{1}{144}(2x+3u)\ln^2(x+\tfrac{1}{2}u)+[\tfrac{25}{324}(x+\tfrac{1}{2}u)+\tfrac{1}{12}Au]\ln(x+\tfrac{1}{2}u)\}, \tag{15}$$

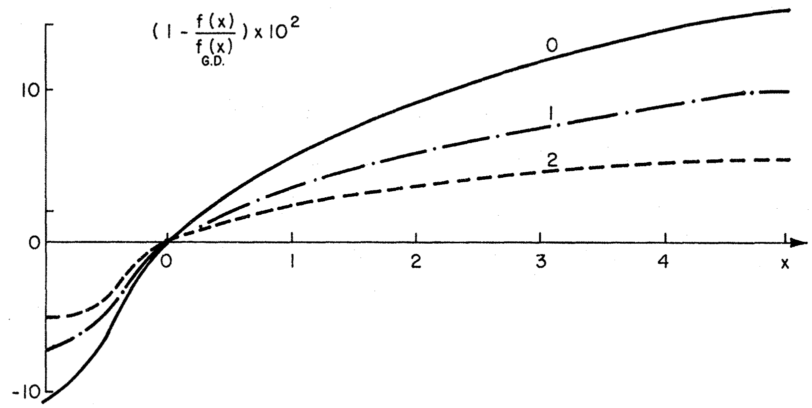

FIG. 1. Plot of $1-f(x)/f_{GD}(x)$ at orders zero, one, and two in ϵ.

where $u=16\pi^2\epsilon/3$ and A is the number defined by Eq. (11).

Detailed fits to experiments for liquid-gas transitions or Ising-like ferromagnets have not been made. The main point has been to verify that the successive corrections of order ϵ and ϵ^2 to mean field theory do go in the right direction. The results are displayed in Fig. 1 and compared with the Gaunt-Domb[6] equation of state

$$H/M^{\delta}=f_{GD}(x)$$

for the three-dimensional Ising model obtained by the use of Padé approximants. The fields and temperature scales have been chosen so that

$$H/M^{\delta}=1 \text{ at } t=0; \quad -t/M^{1/\beta}=1 \text{ at } H=0, \quad t<0. \tag{16}$$

Griffiths[3] has shown that the analyticity of the free energy as a function of M at fixed nonzero t implies that, at infinity, $f(x)$ has an expansion of the form

$$f(x)=\sum_1^{\infty} a_n x^{\gamma-2(n-1)\beta}. \tag{17}$$

This requirement is fulfilled by the right-hand side of Eq. (15) in the framework of the ϵ expansion. For instance the large-x leading terms of this function are

$$f(x)_{x\to\infty}\simeq x+\tfrac{1}{6}\epsilon x\ln x+\tfrac{1}{72}\epsilon^2 x\ln x+\tfrac{25}{324}\epsilon^2 x\ln x$$

which is indeed the ϵ expansion of x^{γ}, where

$$\gamma=1+\tfrac{1}{6}\epsilon+\tfrac{25}{324}\epsilon^2+O(\epsilon^3), \tag{18}$$

in agreement with the value previously obtained in Ref. 1.

It is, of course, possible to replace the form (15) of $f(x)$ by a function equivalent up to order ϵ^2, which displays the correct asymptotic behavior.[7] However, it is preferable to write the equation of state in parametric form in order to fulfill the analyticity condition[8] expressed by Eq. (17). The parametrization of the so-called linear model[9] has been used as follows. Defining the variables R and θ by the relations

$$H=R^{\beta\delta}a\theta(1-\theta^2), \quad t=R(1-b^2\theta), \tag{19}$$

the ϵ expansion of the function $m(\theta)$, where

$$M=R^{\beta}m(\theta), \tag{20}$$

is determined from (15). It is remarkable that, up to order ϵ^2, there are solutions for which the function m is indeed linear[10] in θ (though other choices are possible).

The linear solution

$$m(\theta)=c\theta \tag{21}$$

is valid provided the parameters a, b, and c whose ϵ expansions read

$$a=a_0(1+\epsilon a_1+\epsilon^2 a_2),\quad b^2=b_0^2(1+\epsilon b_1+\epsilon^2 b_2),\quad c=c_0(1+\epsilon c_1+\epsilon^2 c_2),$$

satisfy the following conditions [normalization is again set by Eq. (16)]:

$$a_0=c_0=2^{-1/2},\quad b_0^2=\tfrac{3}{2},\quad a_1-c_1=\tfrac{1}{6}\ln\tfrac{4}{27},\quad c_1-\tfrac{3}{2}b_1=\tfrac{1}{6}\ln 2,$$

and two similar constraints involving also a_2, b_2, and c_2. For simplicity, we display only the particular solution in which c_1 and c_2 are set arbitrarily to zero:

$$a=2^{-1/2}[1+\tfrac{1}{6}\epsilon\ln\tfrac{4}{27}+\epsilon^2(\tfrac{1}{8}\ln^2 2-\tfrac{1}{6}\ln 2\ln 3+\tfrac{1}{8}\ln^2 3-\tfrac{25}{108}\ln 3+\tfrac{25}{162}\ln 2)],$$

$$b^2=\tfrac{3}{2}[1-\tfrac{1}{3}\epsilon\ln 2+\epsilon^2(\tfrac{1}{54}\ln^2 2-\tfrac{25}{486}\ln 2)],$$

but other external requirements could lead to another solution.

Finally, it is interesting to note that in this framework one is naturally led to use the form of equation of state recently suggested by Migdal.[11] It is an implicit equation from which the temperature has been eliminated in favor of the susceptibility χ, or its inverse r. It reads

$$H/r^{\delta/\delta-1}=\varphi(y),\quad y=M/r^{1/\delta-1}. \tag{22}$$

By the same method, the function φ has been determined up to order ϵ^2. The result is

$$\varphi(y)=y-16\epsilon\pi^2 g^{-1}y^3[1+(\tfrac{1}{2}+A)\epsilon]. \tag{23}$$

At this stage, two comments should be made:
(i) Migdal's condition $\beta+\gamma=\frac{3}{2}$ is valid only at order ϵ (and for $n=1$), since

$$\beta+\gamma=\frac{3}{2}+\frac{(n-1)\epsilon}{2(n+8)}+\frac{(n+2)(n^2+26n+54)}{4(n+8)^3}\epsilon^2+O(\epsilon^3).$$

(ii) The function $\varphi(y)$ has indeed a rapidly convergent Taylor expansion since, at order ϵ^2, it is simply a polynomial of third order. However, for $n>1$, Migdal's parametrization cannot be used.[4]

It is a pleasure to thank Professor M. E. Fisher and Professor P. Hohenberg for valuable comments and discussions.

*Research sponsored by the U. S. Atomic Energy Commission under Contract No. AT(11-1)-3072.

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$$f(x)=(x+\tfrac{1}{2}u)^{\gamma}-\tfrac{1}{3}u+\tfrac{1}{6}\epsilon Au+\epsilon^2\times[\tfrac{1}{72}u\ln^2(x+\tfrac{1}{2}u)+\tfrac{1}{12}Au\ln(x+\tfrac{1}{2}u)],$$

agrees with the Gaunt-Domb result to better than the expected uncertainties of both calculations.

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