Parametric Instabilities in Inhomogeneous Media*

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The theory of the three-wave parametric instability for weakly inhomogeneous media is derived with an application to laser pellet irradiation.

We consider a plasma slab whose properties vary with x. It supports three waves described by amplitudes

$$A_i = a_i(x,t)\exp[i(\omega_i t - \vec{k}_i\cdot\vec{x}) - i\int_0^x \Delta k_i(x)\,dx] + \text{c.c.}, \tag{1}$$

with $\Delta k_i(0) = \sum\omega_i = \sum\vec{k}_i = 0$. Thus at $x=0$ the three waves have frequencies ω and local wave numbers $\vec{k}$ which obey the condition for three-wave parametric instability in a homogeneous medium.[1] $\Delta k_i(x)$ is to be determined from the local dispersion relation for a single wave in the inhomogeneous plasma, while the a's are slowly varying coefficients due to the wave interaction. For the sake of definiteness we consider the case with ω, $\vec{k}$, and Δk real, and wave 3 is considered to be the pump wave of fixed amplitude. The motion for small amplitudes is now described by a pair of coupled equations[2]:

$$\partial a_1/\partial t + V_1(\partial a_1/\partial x) = \gamma_0 a_2^* \exp(i\int_0^x \kappa\,dx) \tag{2a}$$

and

$$\partial a_2^*/\partial t + V_2(\partial a_2^*/\partial x) = \gamma_0 a_1 \exp(-i\int_0^x \kappa\,dx), \tag{2b}$$

with V_i the group velocities of the waves, $\kappa = \sum_i \Delta k_i$, and γ_0 the growth rate of mode amplitude for a homogeneous plasma. We consider only the usual unstable case, γ_0^2 real and positive. These equations would hold equally well if the right-hand side were due to the beating of two or more pump waves. Because of the rapid variation of the phase factor it may be considered as the only x-dependent parameter. This will be true when plasma properties vary little over a wavelength. Small wave damping terms could also be introduced without affecting the treatment. Laplace transforming in time with p the Laplace transform variable, neglecting initial values, eliminating a_2^*, and putting

$$a_1 = \exp[i\int_0^x \tfrac{1}{2}\kappa dx - \tfrac{1}{2}p^*(1/V_1 + 1/V_2)x]\psi^*,$$

we easily find

$$\psi'' + \{\tfrac{1}{4}[\kappa + ip(V_1^{-1} - V_2^{-1})]^2 - \tfrac{1}{2}i d\kappa/dx - \gamma_0^2/V_1V_2\}\psi = 0. \tag{3}$$

If this equation has a well-behaved solution with $\mathrm{Re}(p_0) > 0$, then p_0 will correspond to the eigenvalue for a temporally growing mode. If no such p_0 exists, then only finite spatial amplification can occur which we may study by including a source term in Eq. (3) corresponding to spontaneous emission of waves. We note that since κ increases with x the possible asymptotic behaviors at infinity are $a_1^{I} = \exp[(-p^*/V_1)x]$ and $a_1^{II} = \exp[i\int_0^x \kappa\,dx - (p^*/V_2)x]$. Since $\mathrm{Re}(p) > 0$ we see that if $V_1V_2 > 0$ both solutions are badly behaved at either plus or minus infinity and no temporal normal modes are possible. If $V_1 > 0$, $V_2 < 0$ then a well-behaved solution might be possible, namely, a_1^{I} for $x \gg 0$ and a_1^{II} for $x \ll 0$, the question being of course whether the joining conditions near the origin permit this solution. We must now specialize to a particular dependence for κ. As $\kappa = 0$ at $x=0$, the usual case is the linear one, $\kappa = \kappa'(0)x$. We will see that no purely growing modes are possible in this case, and hence we will discuss later the case $\kappa = \kappa''(0)x^2/2$ which might come about either if the interaction region is at a maximum of density or for special cases of the dispersion relations of the waves.

For $\kappa = \kappa' x$, a simple transformation, $\kappa'(0)x + i(p/V_1 - p/V_2) = \kappa'(0)x'$, reduces Eq. (3) to the well-known parabolic cylinder equation. We have seen before that in general no well-behaved solutions exist if $V_1V_2 > 0$, but it is also well known that no solutions going like $\exp[i(\kappa x'^2/4)]$ at $+\infty$ and $\exp[-i(\kappa x'^2/4)]$ at $-\infty$ exist for the parabolic cylinder equation. This will be easily understood when we discuss the WKB analysis below. Hence only spatial amplification is possible. For this purpose we may choose $p=\epsilon$ where ϵ is a small positive number to give the proper behavior at ∞. We also suppose $|\gamma_0^2/V_1V_2|$

$\gg |\kappa'|$ as will be necessary for sizable amplification, allowing the neglect of $d\kappa/dx$ in Eq. (3), which becomes

$$\psi'' + (\tfrac{1}{4}\kappa'^2 x'^2 - \gamma_0^2/V_1V_2)\psi = \delta(x), \tag{4}$$

where for simplicity we have taken the source at $x'=0$.

We consider first the case $V_1, V_2 > 0$. Then from the boundary conditions we know that $\psi = 0$ for $x' < 0$. Beyond the turning point $x_t = 2\gamma_0/\kappa'(V_1V_2)^{1/2}$ the solution is oscillatory while between 0 and x_t it has the approximate form $\sinh[\int_0^x[\gamma_0^2/V_1V_2 - \frac{1}{4}\kappa'^2x^2]^{1/2}dx]$. There is thus a net e-folding by

$$\int_0^{x_t}[\gamma_0^2/V_1V_2 - \tfrac{1}{4}\kappa'^2x^2]^{1/2}\,dx = \tfrac{1}{2}\pi\gamma_0^2/V_1V_2\kappa'.$$

We would have obtained twice the amplification by putting the source at $-x_t$. Finally, since we are usually interested in the e-folding of intensity rather than amplitude we introduce another factor of 2 to get for the peak intensity of oscillations

$$I = I_0 \exp[2\pi(\gamma_0^2/V_1V_2\kappa')], \tag{5}$$

where I_0 is the nondriven, thermal, source intensity. For effective growth then $\gamma_0^2/V_1V_2\kappa' > 1$. In other words, the wave must grow substantially during the time it propagates to the point where the phase mismatch $\int\kappa\, dx$ is substantial.

In the case $V_1V_2 < 0$ the argument is a little more complex. The turning points of the equation are now on the imaginary axis. For small p of order ϵ the real x axis is displaced slightly above the real x' axis and the proper behavior at $\pm\infty$ is obtained by requiring that in the first and second quadrants we have the solutions ψ_I and ψ_{II} which decrease away from the turning point $+ix_t$. Since these are different solutions having a common boundary (the upper imaginary axis), no solution to the homogeneous equation, i.e., no purely growing modes are possible. Returning now to Eq. (3) with the source we may write $\psi = A[\psi_I(x)/\psi_I(0)]$ for $x > 0$; $\psi = A[\psi_{II}(x)/\psi_{II}(0)]$ for $x < 0$ with

$$A = \psi_I(0)\psi_{II}(0)/[\psi_I'(0)\psi_{II}(0) - \psi_{II}'(0)\psi_I(0)].$$

Normalizing ψ_I and ψ_{II} to go like $x^{-1/2}\exp(\pm\kappa' x^2/4)$ for large $|x|$ we may evaluate the Wronskian along the upper imaginary axis to be of order unity. We then find

$$\psi(0)/\psi(x_t) = \exp\{\int_0^{x_t}[\gamma_0^2/|V_1V_2| - \tfrac{1}{4}\kappa'^2y^2]^{1/2}\,dy\}$$

by integrating along the imaginary axis from the turning point. We find again Eq. (5) for the amplification and we conclude in general for the linear case

$$I = I_0 \exp[(2\pi)\gamma_0^2/|V_1V_2\kappa'|]. \tag{6}$$

We turn next to the quadratic case $\kappa = \kappa'' x^2/2$. As discussed earlier the possibility of a normal mode exists only if $V_1V_2 < 0$ and we consider only this case. We will again assume large γ_0 so the term $d\kappa/dx$ in Eq. (3) is negligible. Hence we have

$$\psi'' + \{\tfrac{1}{4}[\tfrac{1}{2}\kappa''x^2 + ip(|V_1^{-1}| + |V_2^{-1}|)]^2 + \gamma_0^2/|V_1V_2|\} = 0. \tag{7}$$

Now, four turning points exist at

$$x = \pm\{(2/\kappa'')[-ip(|V_1^{-1}| + |V_2^{-1}|) \pm 2i\gamma_0(|V_1V_2|)^{-1/2}]\}^{1/2}. \tag{8}$$

We now look for an eigenvalue p such that two of the turning points lie close to $x=0$, i.e.,

$$p = 2\gamma_0[(|V_1V_2|)^{1/2}/(|V_1| + |V_2|)](1-\Delta), \tag{9}$$

with Δ assumed small, thus reducing Eq. (7) to

$$\psi'' + \tfrac{1}{4}\{[\tfrac{1}{2}\kappa''x^2 + 4i\gamma_0(|V_1V_2|)^{-1/2}][\tfrac{1}{2}\kappa''x^2 - 2i\gamma_0\Delta(|V_1V_2|)^{-1/2}]\}\psi = 0. \tag{10}$$

Near the inner turning points this is simply the harmonic oscillator equation whose solution falling off away from these turning points is $\psi = \exp(-\lambda x^2/2)$, with $\lambda = \exp(-i\pi/4)(\gamma_0\kappa'')^{1/2}(|V_1V_2|)^{-1/4}$ and

$$\Delta = \{[\exp(-i\pi/4)]2^{-3/2}\}(\kappa'')^{1/2}(\gamma_0^2/V_1V_2)^{-3/4}. \tag{11}$$

Hence we have indeed found a temporally growing mode in this case for sufficient γ_0, although we have not yet examined the behavior as $|\Delta| \to 1$, where WKB methods are inadequate.

Finally, it is easy to check the validity of our solution by looking at the anti-Stokes lines coming from the inner turning points of Eq. (10) and noting that the regions in which our decaying solutions are valid do in fact contain the real axes for large $|x|$.

As an application we consider the absorption of light from an intense laser by a solid deuterium pellet. As the light is absorbed the outer portions of the pellet are blown off so that the laser sees at nearly normal incidence a plasma profile of increasing density, running from zero to solid density. We may expect various parametric instabilities,[3] leading to possible absorption of the energy when the light reaches a region of sufficient density that frequencies of natural plasma modes become comparable with the laser frequency ω_0.

We consider here the process where the electromagnetic wave drives two Langmuir oscillations with frequencies $\omega_i = [\omega_p^2 + 3k_i^2(T_0/m_e)]^{1/2}$, where $\omega_p^2 = [4\pi n_0(x)e^2/m_e]$. The laser would satisfy the excitation condition $\omega_0 \approx 2\omega_p$ at a region of lower plasma density then that given for other parametric modes ($\omega_0 \approx \omega_p$), and hence might be absorbed before reaching the higher-density plasma.

In a homogeneous plasma the maximum growth rate occurs for plasma oscillation wavelengths somewhat shorter than that of the incoming light, and wave vectors oriented at 45° to that of the light, in the plane determined by its polarization vector and wave vector. The growth rate is given by[4]

$$\gamma_0^2 = \tfrac{1}{2}\pi(e^2/m^2c^3)P, \tag{12}$$

with P the rms incident power per unit area.

It does not appear possible in this case to satisfy $\kappa' = 0$, and so we may now apply Eq. (6) to determine the spatial amplification in the inhomogeneous plasma, obtaining as a condition for excitation

$$\tfrac{2}{3}\pi^2(e/mc^2)^2(mc^2/T_e)PL/\omega_0 \gg 1,$$

where $L = [(1/n_0)(dn_0/dx)]^{-1}$. For the case of a 10-$\mu$m ($CO_2$) laser, expressing L in millimeters, T_e in keV, and P in units of 10^{14} W/cm^2 (1 kJ/mm^2 nsec) the condition becomes $600PL/T \gg 1$, and would seem well satisfiea for parameters suitable for fusion initiation. Thus, at least as judged from the viewpoint of linear theory, the excitation of two-plasma oscillations would appear to be an important absorption mechanism.

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[2]A formal derivation of these physically obvious equations may be made by expanding the plasma dielectric constant in the presence of inhomogeneity and waves as in Ref. 1.

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