[9]The friction coefficient was not measured until 1964, and the divergence of the mutual friction [see G. Ahlers, Phys. Rev. Lett. 22, 54 (1969)] was unknown until 1969.

[10]Ahlers, Ref. 9.

[11]D. M. Sitton and F. Moss, Phys. Rev. Lett. 23, 1090 (1969).

[12]P. M. Roberts and R. J. Donnelly, Proc. Roy. Soc., Ser. A 312, 519 (1969), and references therein.

[13]D. Sitton and F. Moss, in *Proceedings of the Twelfth International Conference on Low Temperature Physics, Kyoto, 1970*, edited by E. Kanda (Keigaku Publishing Co., Tokyo, 1971), p. 109.

[14]B. E. Springett, Phys. Rev. 155, 139 (1967).

[15]Since $v_{L\,\max} = v_s$ and $v_s \ll v$ in this temperature range, the traps are effectively stationary.

[16]P. Murgatroyd, J. Phys. D: Brit. J. Appl. Phys. 3, 151 (1970), and references therein.

[17]Vinen observed the growth and decay of vortex tangles only in transient using the attenuation of second sound.

[18]Since $v \gg v_n$, the term $v\sigma$ is not appreciably dependent on the heat flux.

[19]J. Wilks, *Properties of Liquid and Solid Helium* (Clarendon Press, Oxford, 1968), p. 120.

[20]W. P. Pratt and W. Zimmerman, Phys. Rev. 177, 412 (1969).

[21]R. L. Douglass, Phys. Rev. 141, 192 (1966).

[22]We neglect the self-induced attractive force of a vortex pair, this being on the order 10^2 to 10^3 smaller than $D\langle v_n - v_s\rangle$ for the range of heat flux encountered here.

Possible Origin of the Quantized Vortices in He II

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Based on considerations of the rotating ideal Bose gas it is suggested that the quantized vortex lines can enter the superfluid contained in a rotating bucket through an adiabatic transformation of the ground state.

It is now generally accepted that the behavior of He II in a rotating bucket can be understood by imposing the Onsager-Feynman quantization condition on the superfluid velocity in the Landau two-fluid equations. By purely thermodynamic considerations one then finds that the normal fluid is in solid-body rotation, while for increasing angular velocity ω there appear one, two, three, $\cdots$ quantized vortex lines in the superfluid in *definite* positions and rotating with the normal fluid.[1] In this picture, which accounts at least qualitatively for the experimental results, the successive quantum states of rotation are thermodynamic equilibrium states and should therefore be considered as different *phases*. The fundamental question remains: How can one understand the successive phase transitions? How are the successive vortices produced?

The most widely accepted view is that the quantized vortices are collective excitations of the condensate similar to the phonons and rotons but of much higher energy. The high energy prevents the direct formation by thermal fluctuations,[2] and Iordanski[3] has proposed a nucleation mechanism according to which the origin of the quantized vortices is understood in a similar manner as the formation of liquid drops in the classical vapor-liquid transition. For the appropriate value of ω the vortices grow from small quantized vortex rings and then move to their equilibrium positions by hydrodynamical forces.

Here we would like to propose another, more purely quantum-mechanical explanation for the origin of the vortex lines, which is suggested by considering an ideal Bose gas in a rotating cylindrical bucket. Blatt and Butler[4] have shown that a rotating, ideal Bose gas undergoes phase transitions similar to those occurring in rotating He II. Their main result is shown in Fig. 1, where the total angular momentum Ω is plotted against the angular velocity ω of the bucket. At a series of critical speeds of rotation $\omega_1, \omega_2, \cdots$, the angular momentum increases by $N_0\hbar$, where N_0 is the number of condensed particles. In general,

$$\Omega = \tfrac{1}{2}(N - N_0)mR^2\omega + N_0 l\hbar\ , \qquad (1)$$

where N is the total number of particles, m the mass of a molecule, and R the radius of the bucket.

One easily sees that the contribution of the ex-

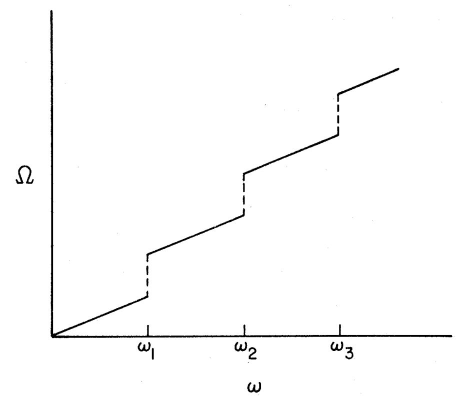

FIG. 1. The angular momentum Ω versus the angular velocity ω of an ideal condensed Bose gas in a rotating cylindrical vessel.

cited particles $(N-N_0)$ is as if they were in solid-body rotation. The centrifugal density distortion is negligible, since ω is assumed to be of order $\hbar/mR^2$. The contribution from the condensate for $\omega_l<\omega<\omega_{l+1}$ is what one expects from a quantized vortex line of strength $l\hbar/m$ on the axis of the bucket. This can be seen as follows: The Bose condensation takes place into the single-particle state, which is the ground state of the Hamiltonian

$$H=H_0-\omega L \tag{2}$$

effective in the rotating frame of reference. In (2) H_0 is the free-particle Hamiltonian and L the angular momentum operator around the axis of the bucket. The eigenvalues of H are

$$E_k=\epsilon_k-\hbar\omega l\ , \tag{3}$$

where k stands for the three quantum numbers n, l, m appropriate to the cylindrical geometry and the ϵ_k are the eigenvalues of H_0. The eigenfunctions are the same as those for the stationary Hamiltonian H_0 and are given by

$$\psi_k=C_kJ_l(k_{nl}r/R)e^{il\theta}\cos(\pi mz/d)\ , \tag{4}$$

where C_k is a normalization constant and d the height of the bucket, and where we have assumed as boundary condition that the normal derivative of ψ is zero at the boundary.[5] If $\omega<\omega_1$ the ground state is $k=(0,0,0)$ and the eigenfunction of the condensed particles (superfluid) is constant. For $\omega=\omega_1$ there is an accidental degeneracy between the states $k=(0,0,0)$ and $k=(1,1,0)$ and for $\omega>\omega_1$ the state $k=(1,1,0)$ becomes the ground state with the wave function

$$\psi_1=C_1J_1(k_{11}r/R)e^{i\theta}\ , \tag{5}$$

where $k_{11}\cong 1.84\cdots$. If one calculates with (5) the probability current density $\vec{J}$ of the ground state, it will have only a θ component. Putting

$$J_\theta=v_\theta\psi_1{}^*\psi_1\ ,$$

one finds

$$v_\theta=\hbar/mr$$

which is the velocity of a vortex on the axis of rotation with circulation h/m.

In this way it goes on. The series of critical velocities ω_l is determined by the increasing series of roots k_{1l} by

$$\omega_l=(\hbar/2mR^2)(k_{1l}{}^2-k_{1,l-1})\ .$$

At every ω_l there is a degeneracy and for higher ω a new lowest state with a higher value of l develops corresponding to a vortex line on the axis with circulation lh/m. This accounts for the discontinuities in Fig. 1.

There is therefore a remarkable similarity with the behavior of rotating He II.[6] In particular, for the Bose gas, just as for He II, there is a first critical value of ω below which the condensate stays at rest. In fact even the values of ω_l are not too different from the observed critical values in He II. One should ask therefore how for the ideal Bose gas the successive vortices are produced, with the hope that this too might give insight into the behavior of He II.

The importance of the ground state is of course a consequence of the Bose statistics and can be justified by considering the canonical partition function $(\beta=1/kT)$

$$Q(N,V,\omega)=\sum_{(n_k)}{}'\exp[-\beta\sum_k n_k(\epsilon_k-\hbar\omega l)] \tag{6}$$

(or for more details the corresponding density matrices), which is assumed to describe the equilibrium state in the rotating frame of reference. Here the summation over the occupation numbers n_k is restricted by the condition $\sum n_k=N$, which is denoted by a prime. One shows that for large N and $V=\pi R^2d$, above a critical density, the ground state is macroscopically occupied by N_0 particles, while the remaining $N-N_0$ particles are distributed over the excited single-particle states and represent the normal fluid in solid-body rotation. It is clear therefore that at the critical angular velocities ω_l, where the ground state is degenerate, the effect of a small perturbation which will break the degeneracy may be crucial. As the degeneracy is a consequence of the cylindrical symmetry, it is of special interest to investigate the

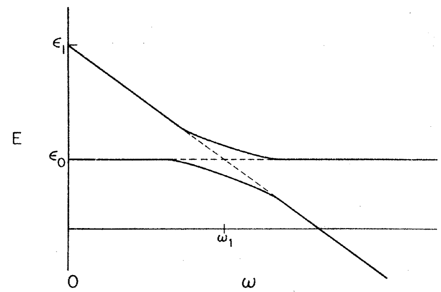

FIG. 2. The effective energies E_0 and E_1 for the two lowest single-particle states without (dashed line) and with (solid line) a "bump" on the wall of the rotating bucket.

effect of a small "bump" on the wall of the rotating bucket.[7] Such a bump can, for instance, be represented by a potential $V(r,\theta)$ which has a small constant value δ in a cylindrical strip along the wall of the bucket (with cross-sectional area A) and is zero otherwise. The effect from first-order perturbation theory on the eigenvalues E_0, E_1 of the two lowest states is shown in Fig. 2. With no perturbation E_0 and E_1 cross (are degenerate) at ω_1, but with the bump they become the solid lines. The ground state is therefore no longer degenerate, and it is important to note that even near ω_1 the splitting between the two levels, although small (because $A\delta \ll \hbar^2/m$), is not thermodynamically small (because V is independent of z).[8] One can show as a result that for large R and d, *only* the lowest state is macroscopically occupied while E_1 merges with the other excited states.

Next let us consider what happens when the bucket is brought from rest *slowly* into rotation. From the adiabatic theorem one can conclude that if ω varies very slowly, the occupation of the energy levels will not change and the wave functions will change continuously. In particular, the wave function for the condensate at rest ($\psi_0 = \text{const}$) will smoothly transform into a wave function which for $\omega \gg \omega_1$ will be close to the wave function ψ_1. Since we saw that ψ_1 corresponds to a quantized vortex on the axis of the bucket, one can say that in this case the vortex enters the fluid by an *adiabatic transformation* of the ground state. This can be seen in more detail from the wave function ψ for the ground state for ω near ω_1. In the approximation where one only includes the two lowest states in the perturbation expansion, one has

$$\psi = (1-f)^{1/2}\psi_0 + f^{1/2}\psi_1 , \tag{7}$$

where f depends on ω and the perturbation such that for $\omega \ll \omega_1$, $f \to 0$, while for $\omega \gg \omega_1$, $f \to 1$. Defining the corresponding velocity field as before, or equivalently by the Madelung relations,[9]

$$\vec{v} = (\hbar/m)\nabla\varphi, \quad \psi = (\psi^*\psi)^{1/2} e^{i\varphi}, \tag{8}$$

one finds that (7) corresponds to an *off-center* quantized vortex line, located at the node of ψ, stationary in the rotating frame of reference and therefore rotating with angular velocity ω in the rest frame. Thus as the bucket is brought slowly into rotation the continuous change of f from 0 to 1 corresponds to the development of a node (vortex) of the ground state at the wall which then moves continuously to the center of the bucket.

This is the picture we propose for the origin of the first quantized vortex. It will be repeated at the successive critical angular velocities ω_2, ω_3, etc. One can also say that at each ω_l a new vortex is formed by *interference* of two weakly coupled, nearly degenerate states, and this shows the similarity with the Josephson effect. The bump which couples the two states plays the same role as the tunnel junction in this effect.

We conclude with three remarks:

(1) From time-dependent perturbation theory it follows that for a sudden change of ω the eigenfunctions will not change. Hence if for $\omega > \omega_1$ one suddenly stops the rotation, the first state E_1 will remain macroscopically occupied, corresponding to a metastable vortex at the center. From our picture one expects therefore that metastable states should be easily observable.[10]

(2) We believe that the qualitative features of our picture of the ground state are independent of perturbation theory and will remain true even for a finite bump.

(3) Finally there is the following qualm. One might think that, instead of (6), one could just as well start from the canonical partition function in the rest frame,

$$Q(N,V,\Omega) = \sum_{(n_k)}{}'' \exp[-\beta \sum_k n_k \epsilon_k], \tag{9}$$

where now the summation over the n_k is restricted by the two conditions $\sum n_k = N$ and $\hbar\sum l n_k = \Omega$. In fact for strict cylindrical symmetry one finds from (9) the *same* results as from the Blatt-Butler Hamiltonian (2) and the partition function (6). However if there is a bump (9) does not have *any* meaning, because a rotating bump corresponds in the rest system to a time-dependent potential, so that one *cannot* speak of definite energy states ϵ_k. We believe that the reason for this dilemma is that *only* for cylindrical symmetry is the equilibrium state stationary *both* in the rest frame and in the rotating frame. However, in general, this equivalence is not necessarily true as the experience with rotating He II has taught us. We believe that *in general one should start from* (6) and if need be transform the macroscopic properties of the equilibrium state back to the rest frame. A precise justification is lacking and would require in our opinion the solution of the problem of the approach to equilibrium.

Finally one should emphasize that the picture we have proposed has been justified so far *only* for the ideal Bose gas. Whether some features of this picture remain valid for the nonideal Bose gas (and therefore for He II) remains to be seen.

*Work supported in part by the National Science Foundation.

[1]A. L. Fetter, Phys. Rev. 153, 285 (1967); G. B. Hess, Phys. Rev. 161, 189 (1967); S. Putterman and G. E. Uhlenbeck, Phys. Fluids 12, 2229 (1969).

[2]For the same reason the vortices once formed will be highly metastable. As Onsager has occasionally remarked, the vortices behave in this respect like the dislocations in solids.

[3]S. V. Iordanski, Zh. Eksp. Teor. Fiz. 48, 708 (1965) [Sov. Phys. JETP 21, 467 (1965)]. The theory was further developed by J. S. Langer and M. E. Fisher [Phys. Rev. Lett. 19, 560 (1967)], and especially by R. J. Donnelly and P. H. Roberts [Phil. Trans. Roy. Soc., London, Ser. A 271, 41 (1971)].

[4]J. M. Blatt and S. T. Butler, Phys. Rev. 100, 476 (1965).

[5]This is the most appropriate boundary condition for an ideal gas, since it makes the ground-state wave function a constant. It corresponds to the periodic boundary condition often used for cubical vessels. The k_{nl} are therefore the roots of the equation $J_l'(x)=0$.

[6]There are of course also differences. In He II the successive vortex lines have all the circulation h/m and they occur in specific positions off center, while for the Bose gas there is always one vortex line at the center with successively increasing circulation. This is perhaps not too surprising because it may be due to the great difference in compressibility of He II and the ideal gas.

[7]One may even argue that such a bump is *necessary*, because only then can there be an exchange of angular momentum between the gas and the vessel, which acts as the angular momentum "reservoir", so that it becomes plausible that one gets an equilibrium state which is stationary in the rotating frame of reference.

[8]For this the boundary condition $\partial\psi/\partial n=0$ at the wall is essential.

[9]E. Madelung, Z. Phys. 40, 322 (1927). Compare also H. Fröhlich, Proc. Phys. Soc., London 87 330 (1966). Such hydrodynamical interpretations are of course only relevant for macroscopically occupied quantum states.

[10]Metastable rotational states of He II have been observed in many experiments. See for instance J. D. Reppy and C. T. Lane, Phys. Rev. 140, A106 (1965).

"Anomalous" Debye-Waller Factor for bcc ^{4}He

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The Debye-Waller factor for bcc ^{4}He is calculated at $T=0$ on the basis of a Hartree-Jastrow ground-state wave function by means of a cluster-expansion technique and is found to be a Gaussian function of the wave-vector transfer to within 1%. We discuss a possible reason for the large deviations from Gaussian behavior observed in recent neutron inelastic-scattering experiments.

In recent neutron inelastic-scattering experiments on the bcc phase of solid ^{4}He, Osgood *et al.*[1] have observed what appear to be anomalously intense one-phonon groups for wave-vector transfers Q between 2.0 and 2.6 Å^{-1}. In this region the sum rule[2]

$$\int_{-\infty}^{\infty}\omega S^{(1)}(\vec{Q},\omega)\,d\omega=(\hbar Q^2/2m)e^{-2W} \qquad (1)$$

is found to be violated by amounts up to a factor of 4 when the conventional Debye model is employed for the Debye-Waller factor.

Werthamer[3] has argued that the Debye model may not be appropriate here and that the sum rule (1) should instead be used to determine an empirical Debye-Waller factor from the first moments of the observed one-phonon groups. The results of his analysis are shown in Fig. 1 together with the Debye model result for which $2W=3\hbar^2Q^2/4mk\Theta$ at $T=0$ and the Debye temperature is taken to be $\Theta=22.5$°K. Werthamer noted that, relative to that for the Debye model, the empirical Debye-Waller factor is not only anom-