Effects of Unitarity on the Multiperipheral Model*

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Two models are presented for which the full multiparticle S matrix is unitary at high energies. The production mechanism is based on the multiperipheral model. It is shown that the elastic-scattering amplitude contains a new type of cut in the angular-momentum plane which is dynamical in origin. This unitarity cut plays a crucial role in enforcing the Froissart bound.

In order to construct a realistic model of diffraction scattering and particle production at high energies it is necessary to take into account the constraints of multiparticle unitarity. In this note we discuss two models for which the full multiparticle S matrix is unitary at high energies.[1] The production mechanism is similar to that of the multiperipheral model to the extent that secondary particles are created and destroyed from chains which are in turn exchanged between the high-energy primary particles. However, in order for the models to satisfy unitarity, it is essential to take into account diagrams in which the secondaries are produced or destroyed from more than one chain. This means that the elastic-scattering amplitude will have contributions from checkerboard diagrams such as the one shown in Fig. 1(b) as well as from the familiar ladder diagrams of Fig. 1(a).

The most striking new feature of these models is the mechanism by which the Froissart bound is enforced. The sum of the checkerboard graphs, whose presence is required by unitarity, gives rise to a square-root branch cut in the angular-momentum plane. It should be emphasized that this unitarity cut is dynamical in origin as opposed to the almost kinematical origin of the familiar Mandelstam cuts, which are also present here, and the Amati-Fubini-Stanghellini cuts. The unitarity cut is not present in any individual diagram. It is associated with a divergence in the perturbation series for the S matrix.

As is well known, the standard multiperipheral and multi-Regge models do not have the constraints of unitarity built in. As a result, they can give rise to a violation of the Froissart bound by having a Regge pole to the right of $l=1$.[2] In the present case it is also possible for the ladder graphs to generate a pole to the right of $l=1$. However, in our solvable model we find that any pole which passes $l=1$ is always on an unphysical sheet because it has passed through the unitarity cut. Thus it is not possible to violate the Froissart bound.[3,4]

In addition to enforcing this bound, the unitarity cut tends to decrease the importance of the multi-Regge region of phase space. For most values of the input parameters in our models, the multi-Regge region yields a small energy decreasing contribution to the total cross section. In our solvable model the leading l-plane singularity arising from the multi-Regge region can reach unity only if the input pole is itself *greater* than one. In this situation the Froissart bound can be saturated.

Let us now turn to the specification of the models to be discussed here. Two types of particles appear in our models. All states contain two nonidentical, spinless "nucleons," plus an arbitrary number of identical "pions." The pions can be created and destroyed, but not the nucleons. As in the eikonal model, it is assumed that the nucleons retain a large fraction of their longitudinal momenta throughout the scattering process.[5] Working in the c.m. system, we take the S-matrix elements to be a function of Y, the rapidity

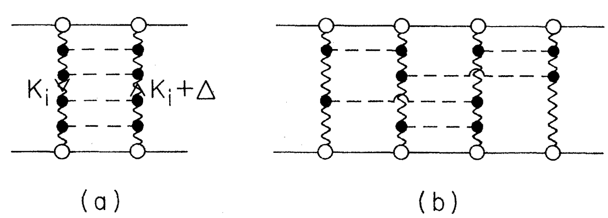

FIG. 1. (a) Typical ladder and (b) checkerboard graphs.

difference between the nucleons; B, the transverse distance between the nucleons; and q_i and y_i, the transverse momentum and rapidity of the ith pion. In the eikonal approximation, the S matrix is diagonal in Y and B. Our model is now completely specified by giving the amplitude for the production of n pions off a single chain, $W_n(Y,\vec{B};\vec{q}_1,y_1,\cdots,\vec{q}_n,y_n)$. By crossing symmetry, W_n also describes chains in which some or all of the pions are incoming. It is convenient to introduce a single operator Z_n which handles all possible production and absorption processes involving n pions. Z_n is related to W_n by

$$Z_n(Y,\vec{B})=\frac{1}{2s}\int\prod_{i=1}^{n}dq_i\,\frac{1}{n!}\,W_n(Y,\vec{B};\cdots):\prod_{i=1}^{n}[a(\vec{q}_i,y_i)+a^\dagger(-\vec{q}_i,y_i)]:;\qquad(1)$$

where a and $a^\dagger$ are the pion creation and annihilation operators normalized such that $[a',a^\dagger]=2(2\pi)^3\times\delta^2(\vec{q}-\vec{q}\,')\delta(y-y')$. The invariant phase-space element is $dq\equiv d^2q\,dy/2(2\pi)^3$, and $s\equiv m^2e^Y$, where m is the nucleon mass. The creation and annihilation operators have been normal ordered in Eq. (1) to prevent a π from being reabsorbed on the same chain from which it was emitted. Since we wish to consider chains from which an arbitrary number of π's are created or destroyed, we introduce a Hermitian operator and unitary S matrix by

$$Z(Y,\vec{B})=\sum_{n=0}^{\infty}Z_n(Y,\vec{B}),\quad S(Y,\vec{B})=e^{iZ(Y,\vec{B})}=\sum_{N=0}^{\infty}\frac{i^N}{N!}Z^N.\qquad(2)$$

Let us start by considering a model which is simple enough to be solved exactly. We take the exchange mechanism between adjacent particles on the chain to be that of a fixed pole, and ignore correlations between transverse momenta. The rapidities are taken to be strongly ordered. Working in the c.m. system, we then write[6]

$$\frac{1}{2s}W_n(Y,\vec{B};\vec{q}_1,y_1,\cdots,\vec{q}_n,y_n)=e^{-Y}f(\vec{B})\prod_{i=0}^{n}\exp[\alpha(y_i-y_{i+1})]\theta(y_i-y_{i+1})\prod_{j=1}^{n}g(\vec{q}_j),\qquad(3)$$

where $y_0=-y_{n+1}=\frac{1}{2}Y$. It is convenient to introduce creation and annihilation operators, $c^\dagger$ and c, defined by

$$c=(\lambda Y)^{-1/2}\int\frac{d^2q}{(2\pi)^2}\int\frac{dy}{4\pi}g(\vec{q})a(\vec{q},y),$$

where the effective coupling constant λ is chosen so that $[c,c^\dagger]=1$. $Z(Y,\vec{B})$ and $S(Y,\vec{B})$ can now be expressed in terms of the coordinate operator $X=2^{-1/2}(c+c^\dagger)$,

$$Z(Y,\vec{B};X)=f(\vec{B})\exp[(\alpha-1-\tfrac{1}{2}\lambda)Y+(2\lambda Y)^{1/2}X].$$

Clearly S is diagonal in this coordinate representation. We shall be primarily interested in elastic scattering, so the matrix element of S is needed between states with no pions:

$$\langle 0|S(Y,\vec{B})|0\rangle=1+\tfrac{1}{2}is^{-1}M_{22}(Y,\vec{B})=\pi^{-1/2}\int_{-\infty}^{\infty}dx\,e^{-x^2}\exp[iZ(Y,\vec{B});x],\qquad(4)$$

where M_{22} is the elastic-scattering amplitude. It is instructive to examine M_{22} in the angular-momentum plane[7]:

$$\int_0^\infty dY\,e^{-lY}M_{22}(Y,\vec{B})=2im^2\{(l-1)^{-1}-G(l)[-if(\vec{B})]^{h(l)}+C(l,\vec{B})\},\qquad(5)$$

where

$$G(l)=\Gamma\{(2/\lambda)^{1/2}[(l-\alpha_c)^{1/2}-(1-\alpha_c)^{1/2}]\}[2\lambda(l-\alpha_c)]^{-1/2},\quad h(l)=(2/\lambda)^{1/2}[(1-\alpha_c)^{1/2}-(l-\alpha_c)^{1/2}],$$

and $C(l,\vec{B})$ is an entire function of l for all values of B provided $\alpha<1+\frac{1}{2}\lambda$. For $\alpha>1+\frac{1}{2}\lambda$, the only singularities on the physical sheet of the l plane are the branch point at α_c and a singularity at $l=1$ of the form

$$M_{22}(l,\vec{B})\underset{l\to1+}{\longrightarrow}2im^2(l-1)^{-1}[if(\vec{B})]^{h(l)}.\qquad(6)$$

The poles exhibited in Eq. (5) have moved onto an unphysical sheet. The position of the new dynamical branch point is $\alpha_c=\alpha-(1-\alpha-\frac{1}{2}\lambda)^2/2\lambda$ for any value of α.

Figure 2 illustrates the l-plane structure for the case $\alpha\leq1+\frac{1}{2}\lambda$. The N-Reggeon exchange am-

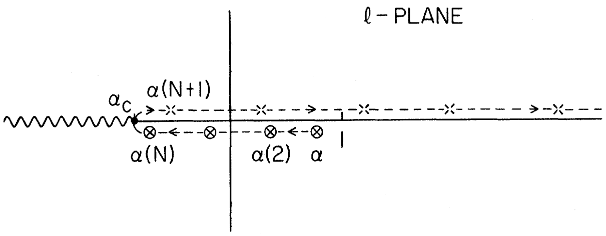

FIG. 2. Analyticity in the l plane. Dashed poles are on the unphysical sheet.

plitude has a pole at $\alpha(N) = 1 + N(\alpha - 1) + \frac{1}{2}\lambda N(N-1)$. The pole at $l = \alpha(1) = \alpha$ is due to the exchange of a single fixed input pole. The pole at $\alpha(2) = 2\alpha - 1 + \lambda$ arises from the ladder graphs, and the poles with $N \geq 3$ from the checkerboard graphs with N vertical lines. In general, $\langle 0 | Z^N | 0 \rangle = f^N(\vec{B}) \exp\{Y \times [\alpha(N) - 1]\}$, so that the series expansion of S given in Eq. (2) diverges. The poles with $N \geq 2$ are dynamical in origin, and the quadratic dependence of $\alpha(N)$ is due to the fact that the number of attractive pairwise interactions increases as $\frac{1}{2}N(N-1)$. The square-root branch point at $l = \alpha_c$ is directly related to the above divergence of the perturbation expansion. Notice that $\alpha_c \leq 1$ for all values of λ and α. The only poles on the physical sheet are those for which $N \leq \overline{N} = (1 - \alpha + \frac{1}{2}\lambda)/\lambda$. Let us imagine that the coupling constant is increased from zero to infinity at a fixed value of $\alpha \leq 1$. For small values of λ the branch point is far to the left in the l plane. As λ is increased the dynamical poles move to the right, but the branch point moves even faster. Each pole eventually collides with the branch point and then moves off onto the unphysical sheet. At $\lambda = 2(1 - \alpha)$ the branch point circles the fixed pole and starts to retreat back to the left. Therefore, for $\lambda > 2(1-\alpha)$ the branch point is the only singularity on the physical sheet.

The behavior of the cross section as the parameters are varied is now easy to follow. The total cross section is dominated by the dynamical pole arising from the ladder graphs as long as this singularity is on the physical sheet. We thus have $\sigma_T(s) \sim (s/m^2)^{2\alpha - 2 + \lambda}$ for $\lambda \leq \frac{2}{3}(1-\alpha)$. On the other hand, for $\lambda > \frac{2}{3}(1-\alpha)$ the branch point dominates and we then find $\sigma_T(s) \sim (s/m^2)^{\alpha_c - 1}[\ln(s/m^2)]^{-1/2}$. Thus, for $\alpha < 1 + \frac{1}{2}\lambda$ the total cross section always goes to zero at high energies. This includes the case $\alpha = 1$, which is given above with $\alpha_c = \frac{1}{8}\lambda$.

For $\alpha > 1 + \frac{1}{2}\lambda$ the scattering amplitude has a branch point at $l = \alpha_c$ and an additional singularity at $l = 1$. To illuminate the form of this singularity we make the particular choice $f(\vec{B}) = e^{-B/R}$. Then the Fourier transform with respect to B gives

$$M_{22}(l, \vec{\Delta}) \simeq 2im^2R_0^2[(l-1)^2 + R_0^2\vec{\Delta}^2]^{-3/2}, \qquad (7)$$

where $R_0 = R(1 - \alpha + \frac{1}{2}\lambda)$, and we have neglected contributions from the branch point at $l = \alpha_c$. The two-dimensional momentum transver is Δ, and at high energies, $t \simeq -\vec{\Delta}^2$. The amplitude in Eq. (7) is just the l-plane singularity associated with scattering from a black disk of radius R_0Y.[4] It gives rise to a total cross section of the form $\sigma_T(s) = 2\pi R_0^2[\ln s/m^2]^2$. If one increases α for fixed λ, the branch points at $l = 1 \pm iR_0\sqrt{-t}$ enter the physical sheet through the unitarity cut when $\alpha = 1 + \frac{1}{2}\lambda$. At this point $\alpha_c = 1$ and $R_0 = 0$. Notice that for sufficiently large values of λ the total cross section always goes to zero at high energies for fixed α.

The unitarity cut, which we have exhibited explicitly in this solvable but quite general model, will be present in a wide class of multiperipheral-like models. In particular, it has been possible to show that the cut exists in a model where W_n coincides with the standard multi-Regge amplitude in the region of phase space in which all subenergies are large.[1] This model cannot be solved analytically; however, it is possible to write down an integral equation for the amplitude $Z_N \equiv \langle 0 | Z^N | 0 \rangle$, which gives the contribution to the elastic scattering amplitude arising from N-Reggeon exchange. The integral equation for Z_N has the same structure as the Lippman-Schwinger equation in two dimensions.[8] The angular-momentum variable plays a role analogous to the energy variable in the nonrelativistic scattering problem. As a result, determining the position of the leading Regge pole that contributes to Z_N is equivalent to solving for the ground state of a two-dimensional N-body system. Since there are $\frac{1}{2}N(N-1)$ attractive two-Reggeon interactions, one expects the leading Regge pole in Z_N to move to the right in the l plane like N^2. In fact, it is possible to prove that this is the case for a wide range of parametrizations of the input Regge pole.[1] Since the S matrix is explicitly unitary, the elastic scattering amplitude cannot have l-plane singularities on the physical sheet to the right of $l = 1$. As a result, it must have a branch cut which is of a different type than those discussed by Mandelstam, and the troublesome poles must be on an unphysical sheet of this cut.[9] This can be shown explicitly in the present model.[1] It is difficult to see how these features could be changed in more sophisticated models that take into account low subenergy effects.

In our solvable model, we find that if the input trajectory is 1 or less, the multi-Regge region of phase space provides a contribution to the total cross section that decreases as a power of the energy. Hence, the experimentally observed constant total cross sections must arise from other sources, such as the fragmentation region or the low subenergy pionization region.

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[1]A more detailed exposition of these models is given elsewhere. S. Auerbach, R. Aviv, R. Blankenbecler, and R. Sugar, Stanford Linear Accelerator Center Report No. SLAC-PUB-1047 (unpublished).

[2]J. Finkelstein and K. Kajante, Phys. Lett. 26B, 305 (1968).

[3]Recently several different approaches have been suggested for enforcing the Froissart bound when the ladder graphs have a singularity to the right of $l=1$ [H. Cheng and T. T. Wu, Phys. Rev. Lett. 24, 1456 (1970); S. J. Chang and T. M. Yan, Phys. Rev. Lett. 25, 1586 (1970); J. Finkelstein and F. Zachariasen, Phys. Lett. 34B, 631 (1971); J. R. Fulco and R. L. Sugar, Phys. Rev. D 5, 1919 (1971)]. However, in these models the Froissart bound is saturated from the multi-Regge region of phase space. This result is unsatisfactory experimentally since particles produced at high energies tend to have rather low relative energies.

[4]See Cheng and Wu, Chang and Yan, Finkelstein and Zachariasen, and Fulco and Sugar, Ref. 3.

[5]Models of this type have been discussed recently by R. Aviv, R. Blankenbecler, and R. Sugar, Phys. Rev. D 5, 3252 (1972); and by G. Calucci, R. Jengo, and C. Reggi, Nuovo Cimento 4A, 330 (1971), and 6A, 601 (1971), and to be published.

[6]In order for the eikonal approximation to be valid one should introduce θ functions to the W_n which restrict the pion rapidities to the range $|y_i| \leq \frac{1}{2}(1-\epsilon)Y$ (see Refs. 1 and 4). Since in most cases ϵ can be set equal to zero at the end of the calculation, we shall not write it explicitly.

[7]A. Erdelyi *et al.*, *Tables of Integral Transforms* (McGraw-Hill, New York, 1954), Vol. I, p. 146.

[8]The Reggeon calculus used here is in the spirit of that discussed by V. N. Gribov, Zh. Eksp. Teor. Fiz. 53, 654 (1967) [Sov. Phys. JETP 28, 414 (1968)]; and H. D. I. Abarbanel, National Accelerator Laboratory Report No. THY-28, 1972 (to be published).

[9]The amplitudes Z_N contain the Mandelstam cuts. In fact the Regge poles enter the physical sheet of the Z_N through these cuts.

Symmetry Relations for Deep-Inelastic Processes*

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Relations between structure functions conventionally obtained from quark-parton or light-cone models are shown to follow from model-independent symmetry assumptions common in hadron scattering. Exotic t-channel exchanges are forbidden. No strong-interaction symmetry beyond isospin is assumed, and SU(3) is used only for vertices involving currents. Relations originally derived for nucleon targets hold for any isospin mirror pair and apply to complex targets appearing in the Mueller formalism for inclusive processes in the target fragmentation region. New relations are derived.

Numerous relations between the electromagnetic and weak structure functions have been obtained[1] by use of either the quark-parton model or the light-cone algebra. We wish to point out that many of the relations can be obtained from general symmetry conditions and are present in a wide class of models,[2] and the same general conditions can be applied to inclusive reactions and lead to new sets of relations. Consider, for instance, the relations[1]

$$4 \geq F_1^{\gamma H}/F_1^{\gamma \tilde{H}} \geq \tfrac{1}{4}, \tag{1a}$$

$$F_1^{\gamma H} + F_1^{\gamma \tilde{H}} \geq \tfrac{5}{18}(F_1^{\nu H} + F_1^{\nu \tilde{H}}), \tag{1b}$$

where H and $\tilde{H}$ are isospin mirror states. These relations deal with the dependence of the scaling function $F_1(\omega = 2M\nu/Q^2)$ on internal symmetry variables alone at fixed values of the energy-