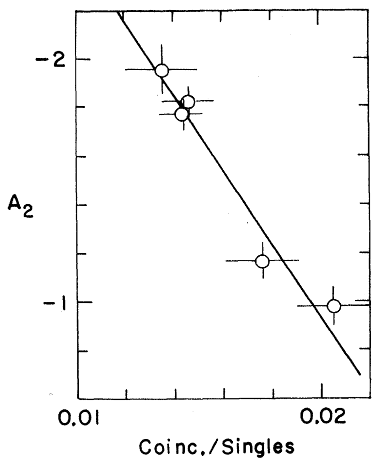

FIG. 1. A_2, as reported by Pattenden and Postma (Ref. 1), versus relative yield of high-energy fragments, as revealed by the coincidence-to-singles ratio.

five values or three degrees of freedom in a bivariate distribution, this corresponds to a significance level (probability that a random bivariate sample would give this result) of much less than 1%. It can be concluded that fission-channel properties are correlated with fragment kinetic energies and thus, presumably, also with $\bar{\nu}$. The effects observed in the present experiment on ^{235}U can be compared with similar work (of considerably higher quality) reported by Felvinci and Melkonian[9] for ^{233}U. The importance of indirect methods such as this for determining fission-channel properties for ^{239}Pu, which does not permit direct determination by nuclear alignment studies, should also be pointed out.

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[1]N. J. Pattenden and H. Postma, Nucl. Phys. A167, 225 (1971).

[2]N. J. Pattenden, private communication (1971).

[3]M. S. Moore, in *Statistical Properties of Nuclei*, edited by J. B. Garg (Plenum, New York, 1972), p. 55.

[4]G. A. Cowan, B. P. Bayhurst, R. J. Prestwood, J. S. Gilmore, and G. W. Knobeloch, Phys. Rev. C 2, 615 (1970).

[5]M. S. Moore and L. G. Miller, in *Proceedings of the Symposium on the Physics and Chemistry of Fission, Salzburg, 1965* (International Atomic Energy Agency, Vienna, Austria, 1965), Vol. I, p. 87.

[6]E. Melkonian and G. K. Mehta, in *Proceedings of the Symposium on the Physics and Chemistry of Fission, Salzburg, 1965* (International Atomic Energy Agency, Vienna, Austria, 1965), Vol. II, p. 355.

[7]S. Weinstein, R. Reed, and R. C. Block, in *Proceedings of the Second International Atomic Energy Symposium on the Physics and Chemistry of Fission, Vienna, Austria, 1969* (International Atomic Energy Agency, Vienna, Austria, 1969), p. 477.

[8]Yu. V. Ryabov, So Don Sik, N. Chikov, and N. Yaneva, Joint Institute for Nuclear Research Report No. P3-5297, 1970 (to be published).

[9]J. P. Felvinci and E. Melkonian, in *Proceedings of the Third Conference on Neutron Cross Sections and Technology, Knoxville, Tennessee, 1971,* edited by R. L. Macklin, CONF-710301 (U. S. AEC Division of Technical Information Extension, Oak Ridge, Tenn., 1971), p. 855.

Determination of Nuclear Spectroscopic Factors by Dispersion Relations*

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The spectroscopic factors for $d \to np$ and $^4\mathrm{He} \to n\,^3\mathrm{He}$ are studied using dispersion relations. It is found that the large distortions represented by the cuts, which have previously prevented dispersion relations from being a practical tool for nuclear physics, can be handled using the conformal mapping techniques of Cutkosky. The method could have general applicability for nuclear physics.

The most accurate determination of coupling constants in particle physics has been achieved via dispersion relations.[1] Soon after they were developed for nucleon-nucleon scattering, forward dispersion relations were derived for composite systems, and application was made to neutron-deuteron scattering[2] for the calculation of spectroscopic factors,[3] the nuclear analog of coupling constants. There have been a number of attempts since then to use dispersion relations in nuclear structure physics,[4] but no successful systematic program has been possible up to now. It is the purpose of the present work to examine the source of past failures and to explore the possibility of a practical program of fixed-energy as well as fixed-angle dispersion relations for nuclear physics.

Recently, Ericson and Locher[5] have reviewed the use of forward dispersion relations in nuclear physics and have carried out calculations with modern data. For neutron-deuteron scattering the results are in satisfactory agreement with known nuclear properties, as were the results of Blankenbecler, Goldberger, and Halpern[2] using earlier data. However, for n-^{4}He scattering the pole approximation is not satisfactory, and the spectroscopic factor for (^{4}He, n^3He) obtained from the forward dispersion relations is quite uncertain. The difficulty, of course, is the treatment of the left-hand cut, which appears from these results to be far more important for the n-^{4}He than for the n-d. We shall try to understand this here.

Since one does not take advantage of the angular distribution with forward dispersion relations, it is likely that one can obtain more accurate results by using dispersion relations in a transfer variable. However, the use of dispersion relations in the $\cos\theta$ variable, if carried out in the pole approximation (as in the successful treatment of the π-nucleon coupling constants[6]), cannot be expected to give accurate results for the pole residues (spectroscopic factors, as explained below) because of the cuts. Here we explore the use of conformal mapping techniques which have been developed by Cutkosky and his collaborators.[7] The mapping which we employ has been shown[7] to enable one to extract coupling constants in the presence of important cut contributions.

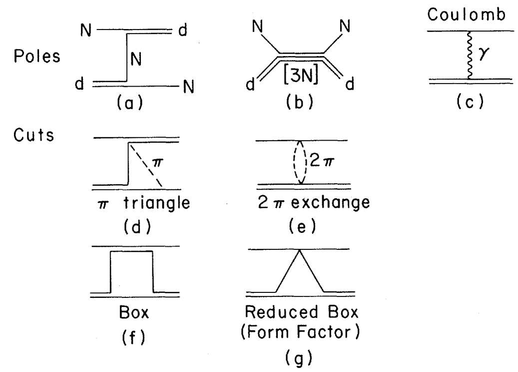

FIG. 1. Diagrams for closest singularities in nucleon-deuteron scattering. Notation is discussed in the text.

Let us first consider nucleon-deuteron scattering. The singularities are illustrated in Fig. 1 for either neutron or proton projectiles. In the figure, N represents a neutron or a proton, and $[3N]$ a ^{3}H or ^{3}He, whichever is appropriate. The Coulomb diagram [Fig. 1(c)] is present only for the p-d case. The once-subtracted forward dispersion relation is

$$\mathrm{Re}(f(E)) = \frac{g_N E}{2\mu E_N(E_N - E)} + \frac{g_{[3N]}E}{2\mu E_{[3N]}(E_{[3N]} - E)} + \frac{k^2}{2\pi^2}\mathrm{P}\int_0^\infty dk'\,\frac{\sigma(k')}{k'^2 - k^2} + \frac{E}{\pi}\mathrm{P}\int_{-\infty}^{E_c} dE'\,\frac{\mathrm{Im}(f(E'))}{E'(E'-E)}, \quad (1)$$

where μ is the reduced mass, $E = (k^2 + M_N^2)^{1/2}$ is the laboratory kinetic energy, σ is the total cross section, and the pole positions E_N and $E_{[3N]}$ are found by setting $u = (p_d - p_{N'})^2 = M_N^2$ and $s = (p_N + p_d)^2 = M_{[3N]}^2$, respectively. The residues g_N and $g_{[3N]}$ are determined by the asymptotic part of the wave functions. For example, for an S-state deuteron with an asymptotic part of the relative wave function given by $\mathfrak{N}e^{-\kappa\chi}/\chi$, $g_N = 2\pi\mathfrak{N}^2$.

The last term in Eq. (1) is the contribution from the left-hand cut, a term neglected in earlier treatments. For nucleon-deuteron scattering the branch point E_c is determined by the π-triangle diagram, Fig. 1(d). Because of the anomalous threshold associated with the loosely bound deuteron, the cut does not start at -70 MeV, which one would naively expect from the u-channel exchange of an additional particle with mass 140 MeV, but at an energy much closer to the physical region. The resulting singularity structure is given in Fig. 2(a).

A simple model of diagram 1(d) is used to learn more about the left-hand cut.[8] It turns out that the contribution of this term to the dispersion relation is not very large. Its influence in determining the spectroscopic factors has been investigated by finding the best fit to the data with and without its inclusion. Taking various cuts in the data,[9] it was found that the spectroscopic factors vary by about 50%, depending upon the range of energies used and that the contribution of the left-hand cut caused a comparable uncertainty. Including the left-hand cut does not stabilize the results as one includes data of increasingly high energy; however, the model for the cut is rather crude

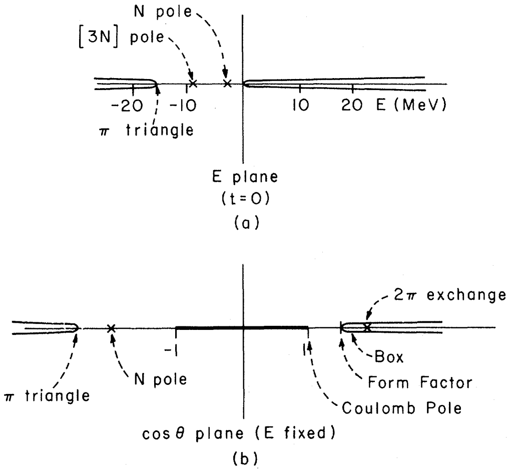

FIG. 2. (a) Singularities for N-d scattering in the E plane for forward scattering; (b) singularities for N-d scattering in the $\cos\theta$ plane for fixed E.

and cannot be expected to be reliable far from the threshold.

The angular dispersion relations have been used in a study of the same data. The singularity structure in the $\cos\theta$ plane is given in Fig. 2(b). The value of the pole position due to nucleon transfer, $\cos\theta_p$, is determined by setting $u = M_N^2$ at each energy. The left-hand branch part is determined from diagram 1(d), and corresponds to the value of the u variable at the left-hand branch point in Fig. 2(a). The right-hand branch point is given by $t=(p_N - p_{N'})^2 = 0.034$ $(\mathrm{GeV}/c)^2$. It corresponds to the deuteron form factor [Fig. 1(d)]. This is also the closest approach of the branch point of the box diagram, as one knows from the fact that the form factor is the reduced box diagram.[10]

Expanding about the nucleon transfer pole for the neutron-deuteron case, one has

$$[4k^4(\cos\theta-\cos\theta_p)^2 d\sigma/d\Omega]_{\cos\theta=\cos\theta_p} = (M_d g_N/\mu)^2 , \qquad (2)$$

where M_d is the deuteron mass and g_N has been defined earlier.

Estimates of g_N have been made using the n-d data at 5.64, 7.01, 9.04, and 14.3 MeV.[9] This was done by introducing a search of the form[5]

$$(d\sigma/d\Omega)(4k^4)(\cos\theta-\cos\theta_p)^2 = \sum_{i=1}^{\mathfrak{N}_p} A(i)(\cos\theta-\cos\theta_p)^i \qquad (3)$$

for the parameters $A(i)$. $A(1)$ is proportional to the spectroscopic factor squared, as seen from Eq. (2).

At each energy a search using the CERN program MINWEE was made with various numbers of parameters, $\mathfrak{N}_p$. It is found that the results are quite sensitive to the value of $\mathfrak{N}_p$. Generally, the optimal $\mathfrak{N}_p$ is one or two larger than the number of partial waves needed to fit the data. The results are summarized in Table I. The results correspond to a value for $g_N \approx 0.12$, which is roughly 25% larger than the value 0.094 which corresponds to the value of $\mathfrak{N}$ obtained from the effective range.

We wish to use this as a test case to study the expansion in a mapped variable to take care of distortions associated with the cuts. Note that some of the distortion normally explicitly introduced in distorted-wave Born-approximation calculations is included in the treatment of the data in the physical region. We map the entire cut onto an ellipse. This is expected to modify the results as if one had approximately included the cuts in the calculation.[7] A search in the form

$$\left(\frac{d\sigma}{d\Omega}\right)(4k^2)[z(\theta)-z_p]^2 = \sum_{i=1}^{\mathfrak{N}_p} B(i)[z(\theta)-z_p]^i , \qquad (4)$$

where $z(\theta)$ is the mapped variable, was made for the $B(i)$.

The nature of the search is the same as the one described above, and the results are given in Table II. Overall, the value of $g_N \approx 0.1$ is found. This result is an improvement of some 20% over the results with the $\cos\theta$ variable. However, since the distortion is small, and there is rough-

TABLE I. Results for N-d elastic scattering using the dispersion relation in $\cos\theta$.

Energy	Optimum $\mathfrak{N}_p$	χ^2	$A(1)$
5.64	3	0.86	0.13
7.01	3,4	0.59,0.47	0.11,0.17
9.04	4	0.48	0.10
14.3	5	1.1	0.2

TABLE II. Results for N-d elastic scattering using dispersion relations in the mapped variable.

Energy	Optimum $\mathfrak{N}_p$	χ^2	$B(1)$
5.64	3	0.85	0.097
7.01	4	0.47	0.097
9.04	4,5	0.5,0.6	0.083,0.093
14.3	5	1.46	0.12

ly a 15% uncertainty in the calculation, it is not possible to state that this is a significant improvement in this case. Since the normalization $\mathfrak{N}$ enters to the fourth power, a careful treatment might select between various potentials. With potentials now in use, there are differences of up to 50% in the value of $\mathfrak{N}^4$ between wave functions with similar values of the D-state probability. The use of Cutkosky's convergence test function[7] would probably allow more definitive conclusions.

Let us now turn to the case of n-^{4}He scattering, where the forward dispersion relation failed. As one can see from Fig. 3, there is one pole in $\cos\theta$, coming from ^{3}He exchange. The left-hand branch point corresponds to $p+d$ exchange (although the π triangle is considerably closer to the physical region than the value given by Ericson and Locher[5]). The right-hand branch point arises from the reduced box diagram, the form factor. Parameter searches of the form (3) and (4) are carried out. In this case, however, the results in the mapped variable differed considerably from those using the $\cos\theta$ expansion. Data at 10 and 15.05 MeV are used.[11] With no mapping it is found that $\mathfrak{N}_p = 6$ is optimum, and that of the various solutions the value of $A(1) \approx 125-150$ is most nearly consistent with the data at the two energies. This value is very nearly the one obtained using the result of the one-pole fit for the forward-dispersion-relation calculation of Ref. 5. The results with the conformal mapping are best for $\mathfrak{N}_p$ either 5 or 6. The values of $B(1) \approx 400-550$ are most satisfactory.

Even with the use of limited data, we have been able to arrive at by far the best determination of this spectroscopic factor which has been achieved, perhaps to within 25% accuracy. Considering the great difficulty in determining this number from

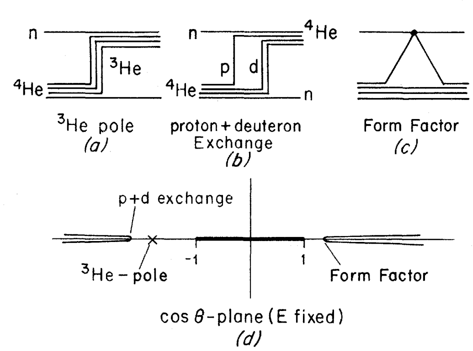

FIG. 3. (a), (b), (c) Diagrams for nearest singularities for n-^{4}He scattering; (d) singularities in the $\cos\theta$ plane at fixed E for n-^{4}He scattering.

n-^{3}He scattering, due to nearby resonances,[12] this is an excellent example of the power of the method to provide an accurate as well as mathematically sound technique for handling distortions.

Thus, first of all one can understand the results of the forward dispersion relation calculations.[5] In the N-d case, it is not that the left-hand cut is far away, for the left-hand cut in the N-d case is actually closer than for the N-^{4}He. It is simply that in the N-d case there is a small discontinuity across the left-hand cut, and so the pole approximation can give reasonable results. However, in the N-^{4}He case the left-hand cut contribution is very large. The exciting and most promising result is that the technique of optimizing the expansion by using a mapped variable seems to be able to handle this large distortion. Details will be given in a forthcoming publication.

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[1]J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. 35, 737 (1963). Contains references to earlier work.

[2]R. Blankenbecler, M. Goldberger, and F. Halpern, Nucl. Phys. 12, 629 (1959).

[3]We define the spectroscopic factor for a composite system as the asymptotic amplitude for breaking into the component parts under consideration. This is not the usual definition.

[4]I. S. Shapiro, in *Interaction of High-Energy Particles with Nuclei, Proceedings of the International School of Physics "Enrico Fermi," Course 38, 1966,* edited by T. E. O. Ericson (Academic, New York, 1967). Contains references to early work.

[5]T. Ericson and M. Locher, Phys. Lett. 27B, 576 (1968), and Nucl. Phys. A148, 1 (1970); M. Locher, Nucl. Phys. B23, 116 (1970).

[6]P. Cziffra and M. J. Moravcsik, Phys. Rev. 116, 226 (1959). An extension for nuclei using pole diagrams with a phase-shift analysis has been carried out by P. E. Shanley [Phys. Rev. Lett. 24, 18 (1970)] and by M. Bolsterli and G. Hale [Phys. Rev. Lett. 28, 1285 (1972)].

[7]R. E. Cutkosky and B. B. Deo, Phys. Rev. Lett. 20, 1272 (1968), and Phys. Rev. 174, 1859 (1968); R. E. Cutkosky, Ann. Phys. (New York) 54, 350 (1969); Y. A. Chao, Phys. Rev. Lett. 25, 309 (1970).

[8]L. S. Kisslinger, in Proceedings of the Fourth Inter-

national Conference on High Energy Physics and Nuclear Structure, Dubna, U.S.S.R., 1971 (to be published). Contains numerical estimate of the left-hand cut contribution.

[9]References in J. D. Seagrave, Los Alamos Report No. LA-DC-10638 (to be published).

[10]R. Eden, P. Landshoff, D. Olive, and J. Polking-Horne, *The Analytic S-Matrix* (Cambridge Univ. Press, Cambridge, England, 1966).

[11]B. Hoop and H. Barschall, Nucl. Phys. 83, 65 (1966).

[12]P. Szydlik and C. Werntz, Phys. Rev. 138B, 866 (1965).

Asymptotic Limit for the Speed of Sound in a System of Relativistically Interacting Particles*

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A relativistic many-body theory is used to evaluate the equation of state to lowest order in the weak coupling constant for a dense system of electrons and neutrinos interacting through the universal Fermi interaction. The asymptotic limit for the speed of sound $v_s \leq c/\sqrt{3}$ is obtained; it is conjectured that a large class of relativistic interactions leads to the same limit.

One of the most important problems in relativistic astrophysics and general relativity is the problem of stability against gravitational collapse for stars in the final stages of evolution.[1] As long as attention is restricted to white dwarfs, a knowledge of nonrelativistic many-body theory and nuclear physics is sufficient.[1] However, when one considers the fate of a star whose mass exceeds the Chandrasekhar mass limit, and asks whether its final state is a neutron star, or inevitable collapse to a black hole, the problem becomes one of relativistic many-body theory and the behavior of matter at asymptotic densities. In the absence of empirical data about matter above nuclear densities, we must rely upon theoretical arguments. One of the simplest, and apparently least dependent upon detailed dynamics, is the conjecture that all realistic relativistic equations of state should lead to a speed of sound $v_s \leq c/\sqrt{3}$.[1] This conjecture met with difficulties, and a less stringent limit $v_c \leq c$ has been suggested.[2] However, none of the arguments leading to either of the above limits is based upon a fully relativistic many-body theory which includes interaction between particles.

We have shown that the limit $v_s \leq c/\sqrt{3}$ is a consequence of at least two relativistic interactions. Our results are based upon a fully relativistic many-body theory.[3] We present at this time the results for a system of neutrinos coupled to a dense degenerate Fermi sea of electrons, with interactions included to lowest order in the weak coupling constant G. The universal Fermi interaction is assumed, and the system is taken to be at zero temperature.

In addition to yielding an asymptotic limit for v_s, the system described above is of importance in neutrino astrophysics,[4,5] and in the case of finite temperatures for some cosmological models.[6] Finally, the relative mathematical simplicity of the model leads to results which (to lowest order) may be expressed in simple analytic form.

Let us restrict our attention to the lowest-order contribution to the interaction $e+\nu \rightarrow e+\nu$ due to the interaction Hamiltonian[7]

$$H_w = \frac{G}{\sqrt{2}} \bar{\psi}_e \gamma^\mu (1-\gamma_5)\psi_\nu \bar{\psi}_\nu \gamma_\mu (1-\gamma_5)\psi_e + \text{H.c.} \quad (1)$$

The system is assumed to contain neutrinos with number density $n_\nu(\beta)$ at temperature $\beta = 1/T$. The density and temperature effects are included through boundary conditions[3] imposed on the neutrino two-point function. The self-energy, corresponding to the Feynman diagram of Fig. 1, is easily shown to be[8]

$$\Sigma(k, \beta) = (G/\sqrt{2})\gamma^0 (1-\gamma_5) n_\nu(k, \beta), \quad (2)$$

with $n_\nu(k, \beta)$ the Fermi-Dirac distribution function for neutrinos. At zero temperature, $n_\nu(k, \beta) \rightarrow k_F^3/6\pi^2$. Next consider the effect of the interaction (1) on a system containing (in addition to neutrinos) electrons of number density $n_e = p_F^3/3\pi^2$ at zero temperature. As a result of the degenerate Fermi sea of neutrinos, the electron pressure will depart from that of an ideal Fermi