Order-Parameter Correlations within the Screening Approximation

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A central problem in the theory of phase transitions is the calculation of the order-parameter correlation function. Here we study the correlation function of an n-component order parameter whose configuration energy is determined by the usual Landau functional. Using a screening approximation, we obtain a simple closed form for the Fourier transform of the correlation function and examine its dependence on wave vector $\vec{k}$ and the reciprocal correlation length κ.

When Landau's theory of phase transitions[1] is extended to include averages over all order-parameter configurations, the partition function can be expressed as the functional integral

$$Z_\Omega = \int \delta\varphi\, e^{-\beta F_\Omega[\varphi]}. \tag{1}$$

Here $\delta\varphi$ denotes an arbitrary configuration of the n-component order parameter in volume Ω, and F_Ω has the form

$$F_\Omega[\varphi] = \int_\Omega d^3x \left[\sum_{i=1}^{n} \left(\frac{a}{2}\varphi_i^2 + \frac{Z_0^{-1}}{2} |\nabla\varphi_i|^2 \right) + \frac{b}{4n}\left(\sum_{i=1} \varphi_i^2 \right)^2 \right], \tag{2}$$

with $a = (T - T_c^0)a'$, and a', b, and Z_0^{-1} are positive constants. In this same scheme, the normalized order-parameter correlation function becomes

$$G(\vec{x}) = \frac{1}{n}\sum_{i,j} \int \delta\varphi\, \varphi_i(\vec{x})\varphi_j(0) \frac{e^{-\beta F_\Omega}}{Z_\Omega} = \int \delta\varphi\, \varphi_i(x)\varphi_i(0) e^{-\beta F_\Omega}/Z_\Omega. \tag{3}$$

When b vanishes, the functional integration reduces to a Gaussian form, and one obtains the Ornstein-Zernike[2] result

$$G_0(\vec{r}) = (Z_0 T/4\pi) e^{-\kappa_0 r}/r. \tag{4}$$

with $\kappa_0 = (aZ_0)^{1/2}$ and $r = |\vec{x}|$.

In treating the effects of the interaction, we introduce the usual diagramatic procedure in which the four-point interaction $(b/4n)(\sum_i \varphi_i^2)^2$ is represented by a dashed line connecting four solid lines each of which represents the Fourier transform of (4),

$$g_0(k) = Z_0 T/(k^2 + \kappa_0^2). \tag{5}$$

All wave-vector sums are cut off at k_D^{-1}, corresponding to the basic cell size for which the Landau form of F is appropriate. Using Dyson's equation, the Fourier transform of the exact correlation function is related to $g_0(k)$ by the self-energy

$$g(k)^{-1} = g_0(k)^{-1} - \Sigma(k^2). \tag{6}$$

A dressed reciprocal correlation length κ can now be introduced by the relation

$$\kappa^2 = \kappa_0^2 - Z_0 T \Sigma(-\kappa^2). \tag{7}$$

Substituting this back into Eq. (6) we have

$$g(k) = \frac{Z_0 T}{k^2 + \kappa^2 - Z_0 T[\Sigma(k^2) - \Sigma(-\kappa^2)]}. \tag{8}$$

In φ^4 theory, $\Sigma(k^2)$ has branch cuts along the $-k^2$ axis beginning at $-(n\kappa)^2$. Here $n = 3, 5, \cdots$ corresponds to the number of intermediate fluctuation lines. Assuming that Σ is analytic elsewhere in the k^2 plane, and using the k_D cutoff, Σ satisfies the dispersion relation

$$\Sigma(k^2) = \frac{1}{\pi}\int_{-k_D^2}^{-9\kappa^2} \frac{\operatorname{Im}\Sigma(k'^2)}{k'^2 - k^2}\, dk'^2. \tag{9}$$

When $\kappa \gg k$, the branch cut in Σ is sufficiently far from the important region of the k^2 plane that the self-energy appearing in Eq. (8) can be expanded about $-\kappa^2$. In lowest order, this gives an Ornstein-Zernike form for g,

$$g(k^2) \cong Z_\psi Z_0 T/(k^2 + \kappa^2), \tag{10}$$

with a renormalized strength parameter

$$Z_\psi^{-1} = 1 - Z_0 T(\partial\Sigma/\partial k^2)|_{k^2 = -\kappa^2}. \tag{11}$$

As the critical point is approached, the branch cuts move toward the origin, and when $k \gtrsim \kappa$, this expansion is no longer useful. Instead, we introduce an effective k-dependent renormalization parameter $Z_{\rm eff}(k, \kappa)$ by the relation

$$g(k) = \frac{Z_0 T}{k^2+\kappa^2 - Z_0 T[\Sigma(k^2) - \Sigma(-\kappa^2)]} = Z_{\rm eff}(k,\kappa) Z_0 T/(k^2+\kappa^2). \tag{12}$$

For $\kappa \gg k$, $Z_{\rm eff}$ reduces to Z_ψ, Eq. (11), while at the critical point $g(k)$ varies as $Z_{\rm eff}(k, 0)k^{-2}$. The function $Z_{\rm eff}(k, \kappa)$ determines how the correlation function changes between these two forms.

From the dispersion relation Eq. (9), it follows that the self-energy can be determined once $\mathrm{Im}\Sigma$ is known. This approach turns out to be analytically convenient. For example, the renormalization parameter Z_ψ can be simply expressed in terms of $\mathrm{Im}\Sigma$,

$$Z_\psi^{-1} = 1 - \frac{Z_0 T}{\pi}\int_{-k_{\rm D}^2}^{-9\kappa^2} \frac{\mathrm{Im}\Sigma(k'^2)}{(k'^2+\kappa^2)^2}\, dk'^2. \tag{13}$$

In addition, we know some properties which any physically meaningful approximation for $\mathrm{Im}\Sigma$ should satisfy: (1) $-\mathrm{Im}\Sigma(k^2+i0^+)$ is a positive function for $k^2 < -(3\kappa)^2$ and vanishes for $k^2 > -(3\kappa)^2$; (2) for $k^2 \ll -k_c^2$, where k_c is a screening wave vector defined below, $\mathrm{Im}\Sigma$ approaches a constant given by the lowest-order perturbation contribution; (3) at the critical point when $\kappa = 0$, we expect[3] that in the range $-k_c^2 \ll k^2 < 0$

$$-Z_0 T_c \mathrm{Im}\Sigma(k^2+i0^+) = \frac{|k|^{2-\eta}}{k_c^{\ \eta}} \sin\tfrac{1}{2}\pi\eta \sim \tfrac{1}{2}\pi\eta\,|k|^2, \tag{14}$$

where the last form is correct to leading order in the anomalous dimension index η first introduced by Fisher.[4]

Now, in general, we cannot expect to evaluate Σ exactly and some type of approximation must be introduced. Previously, on physical grounds, we investigated the screening approximation[5] for the self-energy shown diagramatically in Fig. 1. Following this work, Wilson[6] suggested to us the relevance of our results for the n-component field. Recently, Ma[7] has applied these ideas to analyze the critical exponents for charged and neutral Bose gases. While the original motivation for the screening approximation arose from the construction of the simplest partial resummation which would conform to the three requirements on $\mathrm{Im}\Sigma$ previously listed, it also generates the leading n^{-1} self-energy correction for an n-component field. The n dependence of a diagram is determined by a factor n^{-1} for each interaction

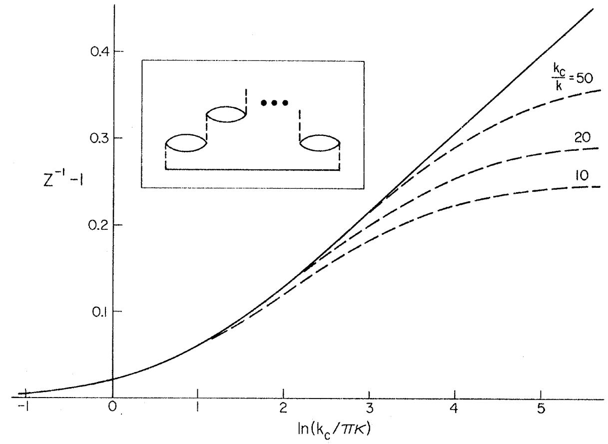

FIG. 1. Inset, self-energy graph used in our screening approximation. Solid line, pole renormalization parameter plotted as $Z_\psi^{-1} - 1$ versus $\ln(k_c/\pi\kappa)$. Dashed lines, $Z_{\rm eff}^{-1} - 1$ versus $\ln(k_c/\pi\kappa)$ for various k_c/k ratios.

and an n for each closed loop. The screening approximation gives a contribution of order n^{-1}, and all other diagrams involve at least one additional power of n^{-1}. Thus, this approximation provides the first nontrivial correction to the spherical model ($n\to\infty$).

The self-energy associated with the diagram of Fig. 1 is

$$\Sigma_s = \left(\frac{\beta b}{n}\right)^2 \int \frac{d^3q}{(2\pi)^3}\,\Pi(q)\,\frac{Z_0 T}{(\vec{q}+\vec{k})^2+\kappa^2}. \tag{15}$$

$\Pi(q)$ corresponds to the infinite bubble sum in Fig. 1, and is given by[8]

$$\Pi(q)^{-1} = \Pi_0(q)^{-1} + \beta b/2n, \tag{16}$$

with

$$\Pi_0(q) = 2n\int e^{ik\cdot x} G_0^{\,2}(x)\, d^3x = [(Z_0T)^2 n/2\pi k]\tan^{-1}(k/2\kappa). \tag{17}$$

In the limit of vanishing κ, $\Pi(k)$ becomes particularly simple,

$$\Pi(k) = \tfrac{1}{4}(Z_0^{\,2}T_c^{\,2}n)(k+k_c)^{-1}. \tag{18}$$

Here the screening length $k_c^{-1} = Z_0^{\,2}T_c b/8$ has been introduced. Thus, in this limit, while Π_0 diverges at low momentum, the bubble sum $\Pi(k)$ is screened by k_c. $\Pi_0(k)$ corresponds to two freely propagating fluctuations and has a spatial correlation function varying as r^{-2}, while the Fourier transform of the bubble sum drops off as r^{-4} for $r \gg k_c^{-1}$. It is this suppression of the long-range part of the pair correlations which leads to a spectral weight which satisfies Eq.

(14).[5] Note that this behavior is not present in the individual perturbation-theory graphs.

Integrating Eq. (15) and analytically continuing it to $-k^2$, we find that

$$-\frac{Z_0T_c}{k_c^2}\,\mathrm{Im}\Sigma_s \cong \frac{4}{\pi n}\left[1-\frac{k_c}{|k|}\tan^{-1}\left(\frac{|k|}{k_c}\right)\right]\left(1-\frac{3\kappa}{|k|}\right)\theta(-(3\kappa)^2-k^2). \tag{19}$$

At $\kappa=0$ this is exact. A direct numerical evaluation shows that it is an excellent approximation for finite κ. Expanding Eq. (19) in powers of k/k_c for $\kappa=0$ gives

$$-(Z_0T_c/k_c^2)\mathrm{Im}\Sigma_s \sim (4/\pi n)\tfrac{1}{3}(k/k_c)^2, \tag{20}$$

so that according to Eq. (14), $\eta=8/(3\pi^2 n)$.

Having obtained an analytic expression for the spectral weight, we turn to the structure of the correlation function. The renormalization parameter Z_ψ obtained using $\mathrm{Im}\Sigma_s$ in Eq. (13) is plotted as the solid line in Fig. 1. Well away from the critical point, where the correlation length is smaller than the screening length ($\kappa \gg k_c$),

$$Z_\psi^{-1} \sim 1+(4\times 1.07/27\pi^2 n)(k_c/\kappa)^2, \tag{21}$$

and the renormalization parameter is essentially 1.[9] However, as the critical point is approached, the correlation length exceeds k_c^{-1} and for $\kappa \ll k_c$[10]

$$Z_\psi^{-1} \sim 1+\eta\left[\ln(k_c/3\kappa)-\tfrac{2}{3}\right] \sim (k_c/\kappa)^\eta. \tag{22}$$

In this region, the renormalized strength of the Ornstein-Zernike form vanishes as a small power η of the reciprocal correlation length.

When the temperature is sufficiently near the critical point that $k \gtrsim \kappa$, the pole approximation for g, Eq. (10), begins to fail. Just as the initial failure of the Ornstein-Zernike relation is described by the parameter η, it is convenient to introduce a "curvature" parameter C_2 to describe the deviation of $g(k)$ from the pole approximation. Expanding the self-energy about $-\kappa^2$ and keeping the second derivative term one has

$$Z_0Tg^{-1} \sim k^2+\kappa^2 - Z_0T\Sigma'(-\kappa^2)(k^2+\kappa^2) - \tfrac{1}{2}Z_0T\Sigma''(-\kappa^2)(k^2+\kappa^2)^2. \tag{23}$$

The curvature coefficient

$$C_2=\kappa^2 Z_0T\Sigma''(-\kappa^2) \tag{24}$$

can be evaluated within the screening approximation, and we find that C_2 is equal to 0.04η.

Eventually, the expansion of g about $-\kappa^2$ is not useful, and the full k,κ dependence of Σ must be taken into account. From the spectral weight Eq. (19) and the dispersion relation Eq. (9), we calculate $\Sigma_s(k^2)$ for k_c large compared to k and κ:

$$Z_0T[\Sigma_s(k^2)-\Sigma_s(-\kappa^2)] = -\eta k^2\left[\ln\left(\frac{\alpha k_c}{[(3\kappa)^2+k^2]^{1/2}}\right)-\frac{3\kappa}{k}\tan^{-1}\left(\frac{k}{3\kappa}\right)+\left(\frac{\kappa}{k}\right)^2\ln\left(\frac{\alpha k_c}{8\kappa}\right)\right]. \tag{25}$$

Here $\alpha=e^{1/3}$. Using this in Eq. (12), $Z_{\mathrm{eff}}(k,\kappa)$ has been evaluated and is plotted (dashed lines) in Fig. 1 as a function of $\ln(k_c/\pi\kappa)$ for various k_c/k ratios. In the limit where $k\to 0$ we recover essentially Z_ψ^{-1}.[11] However, for finite κ the effective renormalization saturates when κ becomes small compared to k, and can be expected to be observable in experiments carried out sufficiently close to the critical point. This saturation reflects the critical-point behavior of $g^{-1}(k)$, and using Eq. (25), we find in the limit $\kappa=0$

$$Z_0Tg^{-1}(k)=k^2+\eta k^2\ln(\alpha k_c/k) \sim k^2(k/k_c)^\eta. \tag{26}$$

Thus the same quantity η which enters into renormalizing the strength of the Ornstein-Zernike pole as T_c is approached also determines the structure of the correlations of the order-parameter fluctuations exactly at T_c.

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[3]Assuming that when $\kappa=0$, $g(k)$ has the homogeneous form $k^{-2}(k/k_c)^\eta$, the spectral weight Eq. (10) is obtained from $\mathrm{Im}\,g(ik)^{-1}$.

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[6]K. Wilson, private communication.

[7]S.-K. Ma, "Critical Exponents for Charged and Neutral Bose Gases Above λ Points" (to be published).

[8]From now on we will replace κ_0 by the renormalized reciprocal correlation length κ in all G_0 Green's functions.

[9]The factor 1.07 in Eq. (21) is $(1-q^{-1}\tan^{-1}q)/2q^2$ evaluated at $q^2=-\frac{1}{9}$. It comes from evaluating the renormalization parameter at $k^2=-\kappa^2$ and would be 1 if Z_ψ^{-1} were evaluated at $k^2=0$.

[10]The power law has, of course, only been generated to leading logarithmic order by the screening approximation. However, the higher-order terms can be obtained by continuing the n^{-1} expansion. These also generate further corrections to η in powers of n^{-1}.

[11]There is a small difference, less than 10%, due to the fact that Z_ψ^{-1} is evaluated for $k^2=-\kappa^2$ rather than $k^2=0$.

Transition from Bulklike Behavior to Josephson-Junction–like Behavior in Superconducting Microbridges

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The behavior of superconducting Sn and Sn-In microbridges has been studied as a function of physical parameters of the evaporated films. For a proper choice of mean free path and bridge size, a regime of ideal Josephson-junction-like behavior appears just below T_c. The data are in good agreement with recent calculations by Baratoff, Blackburn, and Schwartz.

The behavior of superconducting "weak links" has been the subject of several recent experimental[1,2] and theoretical[3-5] studies. Although both dc[1] and ac[2] experiments indicate oscillatory behavior of the phase near the critical temperature T_c, the problem is complicated by the fact that weak links exhibit bulk superconducting properties under some experimental conditions and Josephson-junction–like properties under others. By measuring the temperature dependence of the critical supercurrent of Sn and Sn-In alloy microbridges, we have determined that there is a region of temperature below T_c within which the bridges show near-ideal Josephson-junction–like behavior; this is further supported by simultaneous monitoring of the quality of the ac Josephson effect. The width of this junctionlike regime can be changed by varying the coherence length, either by altering the mean free path of the film via thickness and impurity content, or by changing the temperature. In the latter case, the bridges are shown to make a smooth transition to bulk superconductorlike behavior as the temperature is decreased.

Figures 1 and 2 show the critical current versus temperature data for a number of ~0.5-μm ×0.5-μm superconducting microbridges with varying mean free paths l. For the pure Sn films of Fig. 1, l was decreased by decreasing the film thickness, while for Fig. 2, l was shortened by alloying In into the Sn. The transition temperature T_c was determined by linearly interpolating the critical current to zero from about 20 mK below T_c, which was the highest temperature where the critical current I_c could be determined to ± 20%. This was typically within 5 mK of the temperature at which the dc supercurrent vanished to within the 1 μA resolution of our apparatus. Three theoretical curves are shown in Fig. 1 for comparison with our data. The upper curve is the temperature dependence of the critical current density for an ideal Josephson junction between two identical superconductors,[6,7]

$$J_c(T)=\tfrac{1}{2}\pi R_n^{-1}\Delta(T)\tanh[\Delta(T)/2k_BT], \tag{1}$$

where R_n is the normal-state resistance and $\Delta(T)$ is the superconducting energy gap. At a reduced temperature $(t\equiv T/T_c)>0.95$, J_c is linear in T_c-T for this case. The middle (solid) curve is the critical current of a bulk superconductor, neglecting the current dependence of the gap parameter. Using the temperature dependence of the two-fluid model, the critical current density may be written as

$$J_c(T)=\frac{cH_c(0)}{4\pi\lambda_L(0)}(1-t^2)^{3/2}(1+t^2)^{1/2}, \tag{2}$$