relatively high value of 0.4 at early time to a low value of 0.007 at a later time. This can be at least qualitatively explained by (11) and (9) taking into account the rapid increase of T_- (or v_-) from its initial value as follows: As time proceeds v_d/v_- decreases from its initial value of about 0.5 to a low value of about 0.07 at 120 nsec. From (11) and (9) it is seen that this decrease of v_d/v_- will cause both χ and the wave energy to also decrease. Quantitatively, if we substitute $v_d/v_- = 0.07$ into (11) we find that $\chi = 6.8$, so that $\omega_a^2 \approx 13.6\Gamma^2$. Substituting this value of Γ^2 into (9) with $k^2\lambda_D^2 \approx \frac{1}{2}$ we have

$$\sum_k \langle |\delta E_k|^2 \rangle / 8\pi n T_- \approx 0.003$$

which is in rough agreement with the experimental value of 0.007.

In view of the basic approximation mode, it may be that the rough agreement between the present theory and the mentioned computer simulation and experiments is somewhat fortuitous. However, it does indicate that accounting for the modification of ion orbits by the method of strong turbulence can be important for saturating the ion-acoustic instability. An extended theory to include the wave spectral shape will provide a further test of the present saturation mechanism.

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Universality of Second-Order Phase Transitions: The Scale Factor for the Correlation Length*

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A universality hypothesis relates the amplitude of the singular parts of the coherence length and the specific heat (or surface tension). For the spin-$\frac{1}{2}$ Ising model it is exact in two dimensions and numerically accurate to within 1% in three. It is consistent with measurements on the "Ising-like" systems CO_2, Xe, and β-brass and the "Heisenberg-like" systems $RbMnF_3$ and EuS to within experimental uncertainties (~ 20%). It provides a sensitive and experimentally convenient indicator of symmetry ("universality") class.

The universality hypothesis for critical phenomena[1-4] asserts that the equation of state very near the critical point can be written

$$M(t,h) = (gt)^{\beta} m(nh/(gt)^{\Delta}), \tag{1}$$

where both the critical exponents (β and Δ) and the function m are the same for a whole "universality" class of systems having the same "symmetries." Only the scale factors g and n depend on the details of the particular system considered. Thus, for example, three-dimensional Ising models with different spin magnitudes, lattice structures, and/or (finite) interaction ranges must all be described by the same indices β and Δ, and the same function m and may differ only in the two scale factors g and n. In (1), $t \equiv (T - T_c)/T_c$, M is the magnetization in units of the saturation magnetization, and h is the magnetic field in units of $k_B T/$(magnetic moment per spin). For fluids, $M = (\rho - \rho_c)/\rho_c$ and $h = (\mu - \mu_c)/k_B T$. Some tests of (1) may be inferred from the literature.[5-11]

Direct generalization of the hypothesis (1) to the Fourier transform $\chi(\vec{q}, t, h)$ of the spin-spin correlation function[3] introduces in addition to g

TABLE I. Comparison of X [Eq. (3)] and Y [Eq. (4)] for different universality classes. Experimental values for α, ν, and (the surface-tension exponent) μ are shown in the table; we fit the $d=3$ Ising-like materials using $\alpha=0.125$, $\nu=0.638$, $\mu=1.25$, the Heisenberg-like magnets using $\alpha=-0.13$, $\nu=0.717$. Our hypothesis states that for all systems in the same universality class, X (and Y) is the same.

	$100X$	Y[a]	α_{expt}	ν_{expt}	μ_{expt}
$d=2$ Ising[b]	$100/2\pi$	$(2\pi)^{1/2}$	0	1	1
		Ising-like, $d=3$			
sc Ising[c]	1.651 ± 0.010	⋯	0.125	0.638	⋯
fcc Ising	1.649 ± 0.004	⋯	0.125	0.638	⋯
bcc Ising	1.659 ± 0.006	⋯	0.125	0.638	⋯
β-brass[d]	1.9 ± 0.3	⋯	0.159	0.65	⋯
Xe[e]	2.0 ± 0.5	6.2 ± 0.6	0.08	0.58	1.30
CO_2[f]	1.4 ± 0.2	6.4 ± 0.4	0.125	0.63	1.28
FeF_2[g]	4.8 ± 0.9	⋯	0.16	0.67	⋯
C_6H_{12}-CH_4O[h]	7.1 ± 3.0	41 ± 10	0.125	0.625	1.23
		Heisenberg-like, $d=3$			
fcc Heisenberg[i]	7.6 ± 0.2	0[j]	-0.13	0.717	⋯
EuS[k]	9.1 ± 1.8	⋯	0	0.702	⋯
$RbMnF_3$[l]	6.8 ± 1.4	⋯	0	0.724	⋯

[a] See Ref. 25.

[b] Refs. 16–18, 29; $C_{\text{sq}}/R=(2\pi)[\ln\sqrt{2}+1)]^2\ln(1/t)$; $C_{\text{tr}}/R=(3\sqrt{3}/4\pi)(\ln 3)^2\ln(1/t)$; $a_{\text{sq}}/\xi_{\text{sq}}=2\ln(\sqrt{2}+1)t$; $a_{\text{tr}}/\xi_{\text{tr}}=\sqrt{3}\ln(3)t$; $a_{\text{sq}}\sigma_{\text{sq}}/k_BT_c=-2t\ln(\sqrt{2}+1)$; $a_{\text{tr}}\sigma_{\text{tr}}/k_BT_c=-t\sqrt{3}\ln(3)$ with a the nearest-neighbor distance.

[c] Refs. 3, 20.

[d] Refs. 30, 31.

[e] Refs. 32–35.

[f] Refs. 32, 36–38.

[g] Refs. 39, 40.

[h] Refs. 9, 41.

[i] Ref. 21; $\alpha=-0.14\pm0.06$. With $\alpha=-0.13$, one finds $C/R=(-6.8\pm0.2)t^{-\alpha}$.

[j] "Surface tension" and Y vanish for complete spin symmetry.

[k] Refs. 42, 42; $\alpha=0\pm0.3$.

[l] Refs. 44, 45; $\alpha=0.007\pm0.02$ from thermal expansion, Ref. 46.

and n a length scale l,

$$\chi(\vec{q},t,h)=n(gt)^{-\gamma}f(ql/(gt)^{\nu},nh/(gt)^{\Delta}), \tag{2}$$

where now the function f is by hypothesis universal, i.e., the same for all systems in a given symmetry class. We now develop an *additional* hypothesis which connects n and l, thus reducing the number of independent scale factors from three to two.

The singular part F_s of the free energy per site (per particle) in units of k_BT scales as $F_s \propto n^{-1}\times(gt)^{2-\alpha}$. Thus, the (singular) free energy belonging to a region of volume ξ^d in d dimensions [$\xi \propto l(gt)^{-\nu}$ is the coherence length] scales as $n^{-1}\rho l^d(gt)^{2-\alpha-d\nu}$ (ρ is the site or particle density), which is a temperature-independent number, when the scaling law $d\nu=2-\alpha$ holds. *We propose*[3] *that $\rho l^d/n$ is universal, i.e. (in terms of directly measurable quantities), that the combination*

$$X\equiv t^2\alpha C\xi^d/k_B \text{ for } d\nu=2-\alpha, \tag{3a}$$

$$X\equiv(gt)^{d\nu-2+\alpha}t^2\alpha C\xi^d/k_B \text{ for } d\nu\neq2-\alpha \tag{3b}$$

is the same numerical constant for all systems belonging to the same universality class. $C=At^{-\alpha}/\alpha$ [$\rightarrow -A\ln(t)+\text{const}$, as $\alpha\rightarrow0$] is the leading singular part of the specific heat[12] per unit volume. [Introduction of the factor α in (3) makes X only weakly dependent on α for small α.] Our hypothesis is related to previous suggestions.[6,13-15]

As a check of this hypothesis (3), we have determined X (at $h=0$, $T\rightarrow T_c^+$, where the best data are available) for a variety of different systems, both theoretical and experimental, as shown in Table I. For the $d=2$, spin $=\frac{1}{2}$ Ising model $C\propto-\ln t$, $\xi\propto t^{-1}$, so $d\nu=2-\alpha$ and by a judicious interpretation of the $\alpha\rightarrow0$ limit, $X=-(t\xi)^2C/k_B\ln t$. The specific-heat and correlation-length amplitudes are analytically known for the square (sq) and triangular (tr) lattices,[16-18] and one finds[19] $X=(2\pi)^{-1}$. We have forced the $d=3$ data to be fitted by the universal indices $\alpha=0.125$, $\nu=0.638$ (Ising-like[20]) and $\alpha=-0.13$, $\nu=0.717$ (Heisenberg-like[21]). Since $0\neq d\nu-2+\alpha=0.039$ (Ising) and 0.021 (Heisenberg), the definition (3b) must be used, thus requiring knowledge of g (e.g.,[11] from the coexistence curve). Model results[3,1,7] indicate that the variations of $g^{d\nu-2+\alpha}$ are too small to be detected in Table I, so in practice we have taken $g=1$.

Table I shows good agreement (to within uncertainties) between members of the same (?) universality class except in FeF_2 and the binary mixture cyclohexane-methanol. FeF_2 is an anisotropic Heisenberg antiferromagnet, which should become Ising-like sufficiently close to T_c. It is not accurately known where the crossover[22] should take place, but it is at least suggestive that the experimental X lies intermediate between Ising and Heisenberg values. C_6H_{12}-CH_4O is the more difficult to explain, but the experimental uncertainties entering X are quite large. Other possible reasons for these discrepancies include (i) effects on the amplitudes of Fisher renormalization,[23] (ii) the influence of order-parameter–phonon coupling, (iii) the existence of "hidden" symmetries, which remove these systems from the $d=3$, Ising universality class, or (iv) lattice anisotropies.[19] Ignoring these two exceptions, we conclude that the singular free energy in a region ξ^d is exactly $-k_BT_c\ln(t)/4\pi$ ($d=2$, Ising universality class) and approximately $0.08k_BT_c(gt)^{-0.039}$ ($d=3$, Ising) or $0.25k_BT_c(gt)^{-0.021}$ ($d=3$, Heisenberg).

If X is universal and if the surface tension $\sigma \propto l\rho(-gt)^{\mu}/n$ (interface tension in binary mixtures, Bloch wall energy in Ising ferromagnets) can be evaluated[24] qualitatively correctly as $F_s\rho\xi$, then σ can be estimated from the specific heat alone.[6] Thus, the numerical factor

$$Y \equiv (\sigma/k_BT_c)/(t^2\alpha C/k_B)^{(d-1)/d}, \qquad (4)$$

should be universal. Some values for Y are shown[25] in Table I. $Y=(2\pi)^{1/2}$ exactly for the square[26] and triangular isotropic Ising models. The agreement of Xe and Co_2 is encouraging; but, the binary mixture again shows strong deviations, and older data[27] for H_2O give $Y\approx 5$. For the $d=2$ Ising model and the $d=3$ systems Xe and Co_2, the surface free energies σ in an area ξ^{d-1} are[25] exactly k_BT_c and about $0.4k_BT_c$, respectively.

Universality in the form (1)–(3) is *not* a law of corresponding states like the one following from the Van der Waals equation (P/P_c function of ρ/ρ_c and T/T_c only). Our hypotheses specifically *do not* require that quantities like C/ρ, $\xi\rho^{1/3}$, or $\sigma/k_BT_c\rho^{2/3}$ (ρ is the particle or site density) are the same for all members of a given universality class at fixed $t\equiv T/T_c-1$. These latter statements would be correct if the scale factors n and g were the same for all systems in the same class, which is simply not true.[11] For example, the specific heats per particle in the square and triangular Ising models differ by 1%, while our factor X is exactly the same for both. (In the anisotropic square lattice[19,16] C/ρ even goes to zero for lattice anisotropy $\to\infty$.) In the three-dimensional Ising models the specific heats per particle differ by 4%, whereas X agrees within 1%. Thus, the universality assumption is a necessary and perhaps correct generalization of the law of corresponding states. In this spirit, the strongest tests of our hypotheses are for systems which are quite unlike, i.e., have widely different scale factors n and g. An interesting test could be made if the surface tension of He^3 were known very near T_c, since quantum effects are known from older data (Lielmezs and Watkinson[27]) to affect strongly the quantity $\sigma/k_BT_c\rho^{2/3}$, whereas we expect $\sigma\xi^2/k_BT_c$, X, and Y to be independent of quantum effects. Similarly, measurements of the specific heat of the binary mixture polystyrene-cyclohexane[28] would show whether (contrary to our expectations) X depends on the molecular weight.

Our universality hypothesis provides a simple method for determining experimentally whether different systems belong to the same universality class and/or whether "true" critical behavior is being observed. Its application (e.g., the evaluation of X and Y requires only that one measure two (or three) critical properties at a single, fixed t. By contrast, in order to determine universality class by comparing critical exponents, one needs to make measurements over several decades in t. For example, although the measured critical exponents α and ν of β-brass and FeF_2 agree, our comparison of the amplitudes shows that FeF_2 seems to present special problems.

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Tricritical Exponents and Scaling Fields

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The tricritical behavior of a classical three-well–potential model for two-component systems (such as He^3-He^4 mixtures) is discussed by using renormalization-group techniques. The tricritical exponents and scaling fields are calculated for three dimensions.

A model for a two-component system with a tricritical point (such as in He^3-He^4 mixtures) is found to show molecular-field tricritical behavior (apart from logarithmic corrections). The model can be viewed as a generalization of the model of Blume, Emery, and Griffiths (BEG).[1] It is solved in three dimensions by using Wilson's approximate renormalization-group recursion relations.[2] The recursion relations were previously shown to give nonclassical behavior for ordinary critical points.[2-6] The tricritical transition differs from an ordinary second-order transition by the existence of an additional density with critical fluctuations.[7,8] The tricritical scaling fields are also evaluated. They are the field variables in which the thermodynamic and correlation functions are scaling invariant.[7,9]

We propose a model that exhibits a first-order, second-order, or tricritical phase transition depending on the value of an external nonordering field g.[8] Let $s(\vec{R})$ denote a classical Ising spin of range $-\infty < s < +\infty$ at site $\vec{R}$ of a discrete cubic lattice; and let $\vec{\delta}$ denote all nearest-neighbor lattice vectors. Then we consider a system with the Hamiltonian $-\mathcal{H}/k_B T \equiv H = H_{ex} + H_{pot}$, in which the exchange-interaction term and the potential-energy term are given by

$$H_{ex} = J' \sum_{\vec{R},\vec{\delta}} s(\vec{R}) s(\vec{R}+\vec{\delta}), \tag{1a}$$

and

$$H_{pot} = -\sum_{\vec{R}} [r' s^2(\vec{R}) + u' s^4(\vec{R}) + v' s^6(\vec{R})] \tag{1b}$$

with the coefficients r', u', and v' depending on the temperature T, nonordering field g, and strength of the tricritical interaction [compare Eqs. (3) and (5)].

The *Ansatz* for the potential-energy term H_{pot} can be justified either by starting from a tricritical Landau-Ginzberg form[10] for $H\{s\}$ or by replacing the BEG spin-1 model for He^3-He^4 mixtures[1] by a continuous-spin model with a spin-weight factor. From both approaches one expects a tricritical transition to occur for some H_{pot} consisting of the sum of three-well potentials at each lattice site $\vec{R}$ (in contrast to two-well potentials for ordinary second-order transitions[2]). In fact, the renormalization scheme yields this result for the potential at the tricritical point (T_t, g_t) as shown in Fig. 1. For fixed $g < g_t$ the side wells of the potential dominate and we expect the model to exhibit a second-order transition at a temperature $T_c(g)$, whereas for $g > g_t$ we expect a first-order transition at $T_1(g)$. The three-well potential simulates the competition between two densities, one of which is the ordering density m as in ordinary critical phenomena, and the other is an additional nonordering