Nonlinear Saturation of the Ion-Acoustic Instability

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A nonlinear theory is presented for the saturation of the current-driven ion-acoustic instability. The saturation mechanism is the modification of ion orbits by finite-amplitude waves, leading to ion trapping. The results are quantitatively compared with recent computer simulations of Biskamp and Chodura, and experiments of Hamberger and Jancarik.

Considerable importance has been attached toward understanding the nonlinear saturation of the current-driven ion-acoustic instability.[1-5] Recently, it has become widely accepted that the dominant nonlinear saturation mechanism involves strong modifications of ion orbits by the unstable acoustic waves. However, no systematic calculation of this saturation mechanism has yet appeared, and hence, no quantitative prediction is available for the saturated wave intensities. The purpose of the present communication is to consider the saturation of the ion-acoustic instability by the method of strong turbulence.[6] With this method the modifications of ion orbits by waves can be calculated in a systematic and quantitative fashion. Neglected are alterations of the population of the tail of the ion distribution function. This is justified *a posteriori* by the fact that the time scale for changes of the ion tail is larger than the time scale for saturation of the unstable waves. That is, the waves are nearly saturated before the ion tail is significantly changed by ion acceleration (e.g., Fig. 1 of Ref. 4). A closed set of equations is thus obtained for the saturated wave spectrum in terms of the linear growth rate. The detailed solution of these equations for the angular spectrum is lengthy and will be reserved for a future publication. For present purposes we merely solve, approximately, for the total electric field energy fluctuations. This theoretical level of saturation is found to be in good agreement with the computer simulations of Biskamp and Chodura.[4]

The saturation of the instability occurs in the following fashion. In the linear theory, where the electron temperature T_- is much greater than the ion temperature T_+, the phase velocity of the ion acoustic wave is much greater than the ion thermal velocity and hence very few ions are in resonant interaction with the wave. However, as the amplitude of the fluctuations grow, the waves perturb the ion orbits and act to broaden the resonance, allowing more ions to participate in resonant interaction with the wave. This increases the ion Landau damping of the wave and eventually succeeds in saturating the instability.

We begin with the nonlinear dispersion relation for ion-acoustic waves in a collisionless plasma. This is just the linear dispersion relation suitably modified to account for the perturbation of the ion orbits by the finite amplitude waves[6]:

$$0 = 1 + \frac{i}{k^2\lambda_{D_+}^{\ 2}}\frac{v_+}{\pi^{1/2}}\int d^3\beta\,\vec{k}\cdot\vec{\beta}e^{-\beta^2}R_{\vec{k}}(\vec{\beta}) + \frac{1}{k^2\lambda_{D_-}^{\ 2}}\left(1 + i\pi^{1/2}\frac{(\omega - \vec{k}\cdot\vec{v}_d)}{kv_-}\right), \tag{1}$$

where

$$R_{\vec{k}}(\vec{\beta}) \equiv \int_0^\infty d\tau\,\exp[i(\omega - \vec{k}\cdot\vec{\beta}v_+)\tau - \tfrac{1}{3}\Gamma_{\vec{k}}^{\ 3}(\vec{\beta})\tau^3]. \tag{2}$$

In the above the subscript plus or minus refers to ions or electrons, respectively, $v_\pm \equiv (2T_\pm/m_\pm)^{1/2}$ is the species thermal velocity, $\lambda_{D_\pm}^{\ 2} \equiv T_\pm/4\pi ne^2$ is the Debye length, $\vec{v}_d$ is the electron drift velocity, and $\vec{\beta}$ is the particle velocity of the ions normalized to v_+. The nonlinear effect of the perturbed orbits is contained in the quantity $\Gamma_{\vec{k}}(\vec{\beta})$; and, indeed, in the limit $\Gamma_{\vec{k}}(\vec{\beta}) \to 0$, the usual linear dispersion relation is recovered. The quantity $\Gamma_{\vec{k}}(\vec{\beta})$ is related to the amplitude of the electric field oscillations by the following equation[6]:

$$\Gamma_{\vec{k}}^{\ 3}(\beta) = \frac{e^2}{m^2}\sum_{\vec{k}}\frac{(\vec{k}\cdot\vec{k}')^2}{k'^2}\langle|\delta E_{\vec{k}'}|^2\rangle R_{\vec{k}'}(\vec{\beta}), \tag{3}$$

where $\delta\vec{E}_{\vec{k}}$ is the Fourier amplitude of oscillation of the electric field.

The evaluation of the integral in Eqs. (1) and (2) is complicated by the fact that the quantity Γ is a function of $\vec{\beta}$ (the normalized ion velocity). However, it is found that the integrand in (1) is sharply peaked about a velocity β_c given by

$$\omega - kv_+\beta_c \approx 2\Gamma_k(\beta_c). \tag{4}$$

The β integration can therefore be asymptotically performed by LaPlace's method.[7] Furthermore, it is found that the time integral (2) can be closely approximated by replacing $\frac{1}{3}\Gamma_{\vec{k}}{}^3(\bar{\beta})\tau^3$ by $\frac{1}{2}\Gamma_{\vec{k}}{}^2(\beta)\times\tau^2$.[7] One can demonstrate numerically that this replacement affords an excellent approximation, particularly in the region $\beta \approx \beta_c$.

With these approximations, we can evaluate $R_{\vec{k}'}(\beta)$ and then solve for the real and imaginary parts of (1) to obtain, since ω is real at saturation,

$$\omega_a{}^2 = \frac{k^2}{1+k^2\lambda_{D-}{}^2}\frac{T_-}{m_+}\left[1+\frac{3}{2}\frac{k^2v_+{}^2}{\omega_a{}^2}\right],\quad \left(\frac{kv_+}{\omega}\right)^2<1, \tag{5}$$

$$\frac{T_-}{T_+}\frac{\omega_a k^2 v_+{}^2}{[2\Gamma_{\vec{k}}{}^2(\beta_c)+k^2v_+{}^2]^{3/2}}\exp\left[\frac{-\omega_a{}^2}{2\Gamma_{\vec{k}}{}^2(\beta_c)+k^2v_+{}^2}\right] = \frac{\vec{k}\cdot\vec{v}_d-\omega_a}{kv_-}. \tag{6}$$

Equation (3) becomes

$$\Gamma_k{}^3(\beta_c) = 0.135\frac{e^2}{m^2}\left(\tfrac{1}{2}\pi\right)^{1/2}\sum_{\vec{k}'}\frac{(\vec{k}\cdot\vec{k}')^2}{k'^2}\frac{\langle|\delta E_{\vec{k}'}|^2\rangle}{\Gamma_{\vec{k}'}(\beta_c)}. \tag{7}$$

Note that ω_a is simply the ion-acoustic frequency, slightly shifted by the first-order correction in the temperature ratio T_+/T_-.

Equations (5), (6), and (7) together determine the saturated wave spectrum $\langle|\delta E_{\vec{k}}|^2\rangle$. However, the solution of these equations for the detailed angular spectrum is a lengthy process outside the scope of the present communication. Here we are just interested in the total wave energy $\sum_k\langle|\delta E_k|^2\rangle$. Hence, to solve (5)–(7) for the approximate total wave energy we make use of the fact that the total wave energy is mainly determined by the fastest growing mode and is relatively insensitive to the details of the spectrum —even though the spectrum may be broad. This is true for the ion-acoustic instability and has already been verified for a number of other instabilities in Refs. 7–9. The explanation is that the fastest growing mode necessarily absorbs most of the energy as it grows in amplitude. It is not until after the fastest growing mode is saturated that a great part of its energy is distributed to other modes.[8,9]

Consequently, when saturation is just about to occur, we can approximate the right-hand side of (7) by

$$\Gamma_{\vec{k}}{}^3 = 0.135\left(\tfrac{1}{2}\pi\right)^{1/2}(e^2k^2/m^2\Gamma_{\vec{k}})\sum_{\vec{k}'}\langle|\delta E_{k'}|^2\rangle, \tag{8}$$

where $\vec{k}$ is the fastest growing mode. Solving (8) for the ratio of field fluctuation energy to particle energy we have

$$\sum_{\vec{k}'}\frac{\langle|\delta E_{\vec{k}'}|^2\rangle}{8\pi nT_-} = \frac{7.4}{(2\pi)^{1/2}}\frac{\Gamma_k{}^4 m_+{}^2}{k^4T_-{}^2}k^2\lambda_D{}^2. \tag{9}$$

Equations (5), (6), and (9) now determine the total wave energy as a function of the drift velocity $\vec{v}_d$, temperature, and the fastest growing mode k. From linear theory, the fastest growing mode is approximately given by $(k\lambda_D)^2 = \frac{1}{2}$. For this case (5) and (6) are approximately given by

$$\omega_a{}^2 = \frac{2}{3}\frac{T_-}{m_+}k^2\left(1+4.5\frac{T_+}{T_-}\right), \tag{10}$$

$$\chi^{3/2}e^{-\chi} = \tfrac{1}{3}\left(1+4.5\frac{T_+}{T_-}\right)\frac{\vec{k}\cdot\vec{v}_d-\omega_a}{kv_-}, \tag{11}$$

where $\chi \equiv \omega_a{}^2(2\Gamma^2+k^2v_+{}^2)^{-1}$, so that the total wave energy is determined by (9)–(11)—provided the fastest growing mode is $k^2\lambda_D{}^2 \approx \frac{1}{2}$.

Finally, we apply Eqs. (9)–(11) to the parameters of the computer simulation.[4] Referring to Fig. 1 of Biskamp and Chodura, and assuming that the electron temperature is slowly varying compared to the ion temperature, we observe that that the temperature ratio T_-/T_+ is about 15 at the instant when the fluctuating electric field energy first saturates. We therefore use this value of T_-/T_+ together with the estimate $v_d \approx v_-/2\sqrt{2}$, given by Biskamp and Chodura, in the right-hand side of Eq. (11). Solving (11) for χ and using (10) to eliminate the frequency we obtain the value of $\Gamma_{\vec{k}}{}^2(\beta_c)$ at saturation

$$\Gamma_{\vec{k}}{}^2(\beta_c) = 0.044T_-k^2/m_+. \tag{12}$$

Notice that the nonlinear damping coefficient is about equal to the bounce frequency.

Finally, substituting (10) and (12) into (9) and using $(k\lambda_D)^2 = \frac{1}{2}$, we obtain the result for the ratio of field fluctuation energy to particle energy

$$\sum_k\langle|\delta E_k|^2\rangle/8\pi nT_- = 0.003. \tag{13}$$

This is in good agreement with the computer simulation result of 0.006.

We can also compare (9), (10), and (11) with the experimental results of Hamberger and Jancarik.[10] This comparison must be made with caution since the conditions in their experiments are changing so rapidly. We consider the ion-acoustic regime of the experiment—what Hamberger and Jancarik call regime A—for which the pertinent data for the total wave energy are given in their Table II. We first note that the ratio of wave energy to thermal energy decreases from a relatively high value of 0.4 at early time to a low value of 0.007 at a later time. This can be at least qualitatively explained by (11) and (9) taking into account the rapid increase of T_- (or v_-) from its initial value as follows: As time proceeds v_d/v_- decreases from its initial value of about 0.5 to a low value of about 0.07 at 120 nsec. From (11) and (9) it is seen that this decrease of v_d/v_- will cause both χ and the wave energy to also decrease. Quantitatively, if we substitute $v_d/v_- = 0.07$ into (11) we find that $\chi = 6.8$, so that $\omega_a^2 \approx 13.6\Gamma^2$. Substituting this value of Γ^2 into (9) with $k^2\lambda_D^2 \approx \frac{1}{2}$ we have

$$\sum_k \langle |\delta E_k|^2 \rangle / 8\pi n T_- \approx 0.003$$

which is in rough agreement with the experimental value of 0.007.

In view of the basic approximation mode, it may be that the rough agreement between the present theory and the mentioned computer simulation and experiments is somewhat fortuitous. However, it does indicate that accounting for the modification of ion orbits by the method of strong turbulence can be important for saturating the ion-acoustic instability. An extended theory to include the wave spectral shape will provide a further test of the present saturation mechanism.

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Universality of Second-Order Phase Transitions: The Scale Factor for the Correlation Length*

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A universality hypothesis relates the amplitude of the singular parts of the coherence length and the specific heat (or surface tension). For the spin-$\frac{1}{2}$ Ising model it is exact in two dimensions and numerically accurate to within 1% in three. It is consistent with measurements on the "Ising-like" systems CO_2, Xe, and β-brass and the "Heisenberg-like" systems $RbMnF_3$ and EuS to within experimental uncertainties (~ 20%). It provides a sensitive and experimentally convenient indicator of symmetry ("universality") class.

The universality hypothesis for critical phenomena[1-4] asserts that the equation of state very near the critical point can be written

$$M(t,h) = (gt)^\beta m(nh/(gt)^\Delta), \qquad (1)$$

where both the critical exponents (β and Δ) and the function m are the same for a whole "universality" class of systems having the same "symmetries." Only the scale factors g and n depend on the details of the particular system considered. Thus, for example, three-dimensional Ising models with different spin magnitudes, lattice structures, and/or (finite) interaction ranges must all be described by the same indices β and Δ, and the same function m and may differ only in the two scale factors g and n. In (1), $t \equiv (T - T_c)/T_c$, M is the magnetization in units of the saturation magnetization, and h is the magnetic field in units of k_BT/(magnetic moment per spin). For fluids, $M = (\rho - \rho_c)/\rho_c$ and $h = (\mu - \mu_c)/k_BT$. Some tests of (1) may be inferred from the literature.[5-11]

Direct generalization of the hypothesis (1) to the Fourier transform $\chi(\vec{q}, t, h)$ of the spin-spin correlation function[3] introduces in addition to g