tently somewhat smaller in magnitude than that predicted by theory, agreement between theory and numerical experiment is within about 5%. Similar agreement is found for comparison of the phases.[7]

As a direct result of observations of anomalous growth of electrostatic longitudinal modes in computer simulations of the Weibel instability, a second-order analysis was done which describes the coupling of two purely growing transverse magnetostatic modes to drive a longitudinal electrostatic mode. A direct comparison of numerical experiment and second-order theory gives agreement within 5% for both the magnitude and phase of the coupling constants.

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Instability of a Relativistically Strong Electromagnetic Wave of Circular Polarization*

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We demonstrate that a circularly polarized electromagnetic wave propagating through a cold plasma and strong enough to make electrons relativistic ($\nu_0 \equiv eE_0/m_e c\omega_0 \gtrsim 1$) is subject to a new pinching type of instability. The linearized equations are solved exactly, without the necessity of introducing an infinite determinant. This allows our results to describe the transition from nonrelativistic to extreme-relativistic electron motion. In the relativistic limit, the growth rates are as high as $0.47\omega_0$.

We describe a new model to study the linear stability of a relativistically strong electromagnetic plasma wave with circular polarization. The treatment is unusual in that it applies to a wide range of strengths of the large-amplitude driving wave. This is because we are able to solve the linearized equations exactly, rather than by approximate truncation of an infinite determinant. The unstable mode is qualitatively different from those found in previous studies of weaker linearly polarized waves[1,2]: It results in a pinching of azimuthal current in the plasma, and is purely growing when the driver has infinite wavelength. Furthermore, the instability has a very large growth rate ($0.47\omega_0$) when the driver is strong enough to make plasma electrons relativistic.

Let us start with a uniform infinite-medium model of a cold, dissipationless plasma containing a large-amplitude circularly polarized driving wave $\vec{E}_0 = E_0(\hat{x}\cos\omega_0 t + \hat{y}\sin\omega_0 t)$, strong enough to make electrons, but not ions, relativistic. In terms of the dimensionless strength parameter of the driver wave $\nu_0 \equiv eE_0/m_e c\omega_0$, this condition corresponds to $\nu_0 \ll M_i/m_e$. The driver is at first considered to be near its reflection point so that its wave vector $\vec{k}_0$ and magnetic field are zero. (Finite-wavelength effects are taken up later.) From the self-consistent equilibrium of a circularly polarized wave,[3-5] it can be shown that

$$k_0^2c^2/\omega_0^2 = 1 - \omega_{pe}^2\omega_0^{-2}(1+\nu_0^2)^{-1/2},$$

a relation which we shall assume throughout the treatment which follows. The background ions form an infinitely heavy uniform medium for most (but not all) of this work.

An uncertainty arises in connection with the dc magnetic field generated by the circular motion of the plasma electrons. This phenomenon, known as the inverse Faraday effect, has been documented experimentally[6] and discussed theoretically by Steiger and Woods.[7] Because the circular motion of the electrons is analogous to a magnetization density, the actual value of this magnetic field depends on the geometry of the plasma, the maximum value being reached for solenoidal geometries. The uncertainty arises because one can consider an infinite medium as the limit of either a long thin solenoid or a flat pancake.

We have neglected the inverse Faraday magnetic field. Two arguments can be presented to justify this neglect. First, Fig. 5 of Steiger and Woods[7] shows that even the maximum magnetic field has a small effect on a self-consistent circularly polarized wave. Second, the characteristic time for deformation of a medium due to the gradient of the inverse Faraday magnetic pressure becomes long compared to the instability growth times derived below as the medium becomes large. The reason is that the inverse Faraday magnetic field will produce at most a finite pressure drop while the inertia tends to infinity. Third, while a definitive study of the inverse Faraday magnetic field awaits further work, the uncertainty of its value does not affect the conclusions regarding stability presented here (except for factors of order unity). There are two reasons for this statement: First the dc field does not qualitatively change the zero-order electron motion from circles, and the relative magnitude of zero-order electric and magnetic fields (Fig. 5 of Ref. 7). Second, because the important perturbed velocities are along the inverse Faraday field, this field does not change the dynamics of the unstable perturbation.

The wave vector of the perturbation is assumed perpendicular to $\vec{E}_0$, and so linearized quantities have space dependence $\exp(ikz)$. In the lab frame, we linearize the electron velocity and Lorentz factor as $\vec{v}_{tot}=\vec{v}_0+\vec{v}_1$,

$$\gamma_{tot}=\gamma_0+\gamma_1=(1+\nu_0^2)^{1/2}[1+(1+\nu_0^2)(\vec{v}_0\cdot\vec{v}_1)c^{-2}].$$

The basic equations we shall use are the electron continuity equation, the relativistic Lorentz force equation for electrons, and the wave equations for vector and scalar potentials in the Coulomb gauge. The wave equations for the potentials are

$$\nabla^2\vec{A}-c^{-2}\partial^2\vec{A}/\partial t^2=-4\pi c^{-1}ne\vec{v}+c^{-1}\nabla\partial\varphi/\partial t,$$

$$\nabla^2\varphi=-4\pi e(n-n_0),\quad \nabla\cdot\vec{A}=0.$$

We solve the linearized equations in a modified form, derived according to the following steps. Into the x and y components of the wave equation for $\vec{A}$, substitute for v_{1x} and v_{1y} expressions obtained from the Lorentz force equation. These transverse components of the force equation express conservation of canonical momentum in first order:

$$\vec{P}_1\equiv m_e(\gamma_0\vec{v}_{1\perp}+\gamma_1\vec{v}_0)+e\vec{A}_1/c=0.$$

Substitute for the scalar potential $4\pi en_1/k^2$. Next, take the time derivative of the continuity equation, and substitute for dv_{1z}/dt the expression obtained from the z component of the Lorentz force equation. The resulting system of coupled linearized equations is

$$d^2A_{1x}/dt^2+k^2c^2A_{1x}+\omega_{pe}^2(1+\nu_0^2)^{-3/2}[A_{1x}(1+\nu_0^2\cos^2\omega_0t)+A_{1y}(\nu_0^2\cos\omega_0t\sin\omega_0t)]=4\pi ec^2\nu_0(1+\nu_0^2)^{-1/2}n_1\sin\omega_0t, \tag{1}$$

$$d^2A_{1y}/dt^2+k^2c^2A_{1y}+\omega_{pe}^2(1+\nu_0^2)^{-3/2}[A_{1x}(\nu_0^2\cos\omega_0t\sin\omega_0t)+A_{1y}(1+\nu_0^2\sin^2\omega_0t)]=-4\pi ec^2\nu_0(1+\nu_0^2)^{-1/2}n_1\cos\omega_0t, \tag{2}$$

$$d^2n_1/dt^2+\omega_{pe}^2(1+\nu_0^2)^{-1/2}n_1=en_0k^2\nu_0m^{-1}(1+\nu_0^2)^{-1}(A_{1x}\sin\omega_0t-A_{1y}\cos\omega_0t). \tag{3}$$

Equations (1) and (2) together are the curl$\vec{B}_1$ Maxwell equation, since $k^2c^2\vec{A}_1=c^2\,\mathrm{curl}\vec{B}_1$. The $d^2\vec{A}_1/dt^2$ terms represent the displacement current, the last term on the left side is proportional to the current $n_0\vec{v}_1$, and the driving term is the current $n_1\vec{v}_0$. Equation (3) is the time derivative of the electron continuity equation. The last term on the left side is proportional to the electrostatic restoring force eE_{1z}, while the driving term represents the $\vec{v}_0\times\vec{B}_1$ force on the electrons. We emphasize that the terms which drive this system are the current $n_1\vec{v}_0$ and the $\vec{v}_0\times\vec{B}_1$ force on the electrons.

To find the exact dispersion relation for Eqs. (1)–(3) we introduce two new dependent variables for the vector potential which rotate with the circularly polarized driver. These new variables are $\alpha_1\equiv A_{1x}\cos\omega_0t+A_{1y}\sin\omega_0t$, and $\beta_1\equiv A_{1x}\sin\omega_0t-A_{1y}\cos\omega_0t$. The third dependent variable is still n_1, the perturbed electron density. When Eqs. (1)–(3) are rewritten in terms of α_1, β_1, and n_1, the new equations have constant coefficients; all the periodic coefficents are eliminated by the change of variables. Solutions for the new system therefore are proportional to the exponentials $\exp(-i\omega t)$, and the condition for a nontrivial solution gives a dispersion relation in the form of a three-by-three determinant:

$$\begin{vmatrix} k^2c^2+\omega_{pe}^2(1+\nu_0^2)^{-1/2}-\omega^2-\omega_0^2 & -2i\omega\omega_0 & 0 \\ 2i\omega\omega_0 & k^2c^2+\omega_{pe}^2(1+\nu_0^2)^{-3/2}-\omega^2-\omega_0^2 & -\omega_{pe}^2(1+\nu_0^2)^{-1/2} \\ 0 & -k^2c^2\nu_0^2(1+\nu_0^2)^{-1} & \omega_{pe}^2(1+\nu_0^2)^{-1/2}-\omega^2 \end{vmatrix}=0. \tag{4}$$

This dispersion relation is a sixth-degree polynomial in ω; its roots are either pure real or pure imaginary. Hence the unstable mode is purely growing for a driver of infinite wavelength.

Asymptotic analytic expressions for the maximum growth rate and most unstable wave number, assuming an infinite-wavelength circularly polarized driver, can be obtained from Eq. (4) in the two limits $\nu_0 \gg 1$ and $\nu_0 \ll 1$. Define a dimensionless growth rate and wave number by $\Gamma \equiv -i\omega/\omega_0$, $x \equiv k^2c^2/\omega_0^2$. For $\nu_0 \gg 1$, Eq. (4) then simplifies to

$$\Gamma^6 + \Gamma^4(2x+4) + \Gamma^2(x^2+3) - x = 0. \qquad (5)$$

Equation (5) indicates that the growth rate is independent of ν_0. The maximum growth rate is $\Gamma_{max} = 0.47\omega_0$, with the most unstable x equal to 2.0. A good approximation to this result can be derived by simply retaining only the last two terms in Eq. (5), yielding $\Gamma_{max} = 0.54$, with the most unstable $x = \sqrt{3}$.

In the opposite limit $\nu_0 \ll 1$, corresponding to a weak driving field, dispersion relation (4) becomes

$$\Gamma^6 + 5\Gamma^4 + 4\Gamma^2 + x^2 - \nu_0^2 x = 0$$

for values of x small compared to unity. In this case the maximum growth rate is $\Gamma_{max} = \nu_0^2/4$, and the most unstable wave number is $x = \nu_0^2/2$.

Figure 1 shows the maximum growth rates given by the numerical solution of Eq. (4). The two asymptotic results derived above are apparent. The solid curve in Fig. 2 shows the dependence of growth rate on wave number for $\nu_0 = 100$, if ions are assumed infinitely heavy. For $\nu_0 \gg 1$ growth rates at large and small x can be derived analytically:

$$\Gamma^2 = x/3 \text{ for } x \ll 1,$$
$$= [(\nu_0^2/x) - 1](1+\nu_0^2)^{-1} \text{ for } x \gg 1.$$

The second expression shows that for very small wavelengths of the perturbation, the plasma simply undergoes stable oscillations.

The physical mechanism for instability is the following: An initial perturbation n_1 gives rise to the current $J_{1\perp} = n_1\vec{v}_0 e$, which produces a perturbed magnetic field $\vec{B}_1$. Thus there is a force on the electrons equal to $e(E_{1z}\hat{z} + c^{-1}\vec{v}_0 \times \vec{B}_1)$ acting in the z direction. The resulting z motion acts back on the original density perturbation through the equation of continuity. Whenever the $\vec{v}_0 \times \vec{B}_1$ magnetic force overcomes the electrostatic restoring force E_{1z}, instability results. If ions as well as electrons are allowed to move in the perturbation, the result is to reduce the restoring force E_{1z} on the electrons, and hence to raise the growth rate. The magnitude of this effect is shown in Fig. 2 which presents the normalized growth rate as a function of perturbation wave number for infinite and hydrogen mass ratios. In both cases the instability produces circularly polarized transverse electric and magnetic perturbations, as well as density perturbations and longitudinal electrostatic fields.

When ions are allowed to move in the perturbed fields, the growth rate of small-wavelength perturbations for $\nu_0 \gg 1$ and $\vec{k}_0 = 0$ stays constant at nearly the maximum value. This is in marked contrast to the fixed-ion case (see the right-hand side of Fig. 2) where the instability disappears for small perturbation wavelengths. For moving ions the asymptotic growth rate for large x

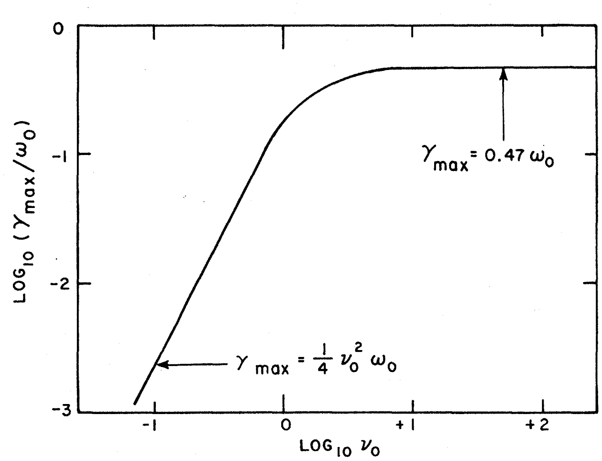

FIG. 1. Dependence of the maximum growth rate γ_{max} on the strength ν_0 of the driving wave. The frequency of the driver is ω_0.

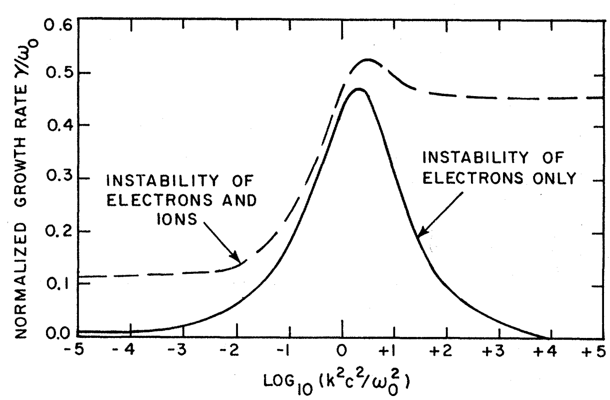

FIG. 2. Dependence of growth rate γ on the wave number k of the perturbation, for a fixed driver strength $\nu_0 = 100$. The frequency of the driver is ω_0. The solid curve shows the growth rate for ions of infinite mass, while the dashed curve is for hydrogen ions.

$\equiv k_0^2c^2/\omega_0^2$ is given by

$$\gamma \cong \omega_0(\nu_0 m_e/M_i)^{1/4}$$

for

$$x \gg [\nu_0^{-2} + (\nu_0 m_e/M_i)]^{-1},$$
$$(M_i/4m_e)^{1/5} \ll \nu_0 \ll M_i/m_e.$$

In regions of lower density where a strong driving wave has a finite wavelength and $\nu_0 \gg 1$, a transformation to rotating variables similar to the one described above allows exact solution of the linearized equations.[8] Details of the calculation will be published elsewhere. The important results can be summaried as follows: In the limit of low plasma densities where $\epsilon \equiv (\omega_{pe}^2/\omega_0^2)(1+\nu_0^2)^{-1/2}$ is small compared to unity, when the driver has nearly vacuum wavelength, the maximum growth rate with fixed ions occurs for $|k| \simeq 2k_0$. The maximum growth rate is $\gamma = \frac{1}{2}\sqrt{3} \times \epsilon^{1/3}\omega_0$, for $\epsilon \ll 1$. By unfolding the transformation of linearized quantities to the rotating coordinate system, the unstable modes can be shown to lowest order in ϵ to consist of (1) a forward-propagating plasma oscillation with frequency $\frac{1}{2}\epsilon^{1/3}\omega_0$ and wave number $2k_0$, and (2) a backward propagating circularly polarized electromagnetic wave with frequency $\omega_0(1-\epsilon^{1/3}/2)$ and wave number $-k_0$. These frequencies and wave numbers obey the matching conditions for three-wave decay instabilities.[1]

In the case $\nu_0 \ll 1$, the pinching instabilities discussed in this Letter must compete with the familiar electrostatic parametric instability, for which the maximum growth rate is[9] $\gamma_{\max} \sim (m_e/M_i)^{1/3}\omega_{pe}$. The electromagnetic pinching instability with $\vec{k}_0 = 0$ and $M_i = \infty$ was shown above to have a maximum growth rate $\gamma_{\max} = \frac{1}{4}\nu_0^2\omega_0$ in this regime. Hence it will dominate the parametric instability if $\nu_0 \gtrsim 0.6$ (for a hydrogen plasma). For $\nu_0 \ll 1$ the effects of pressure are not negligible in general, since the destabilizing $\vec{v}_0 \times \vec{B}_1$ force is opposed by pressure gradients.

The results presented here indicate that considerable anomalous absorption of strong electromagnetic waves may occur when the waves are near their reflection point. The instability is capable of converting the relativistic energy of the driving wave into particle motions. For example, the perturbed electrostatic field can accelerate or trap electrons, because its phase velocity is zero when the driver is at its reflection point, and less than c when the driver has finite wavelength. Furthermore, large energy densities in the form of perturbed magnetic fields are expected to be produced.

In regions of lower density, the growth of backward propagating electromagnetic waves may cause an anomalous reflection of a strong electromagnetic wave.

This new instability may have interesting implications for the two main applications of strong electromagnetic waves, namely, the laser-plasma interaction and pulsar environments, if rotating neutron stars emit strong electromagnetic radiation at their rotation frequency. In the case of the pulsar centered in the Crab nebula, for example, electrostatic fields produced by this type of instability may provide a way to accelerate particles in the nebula. This electrostatic acceleration might complement other acceleration mechanisms proposed for the Crab,[10] which rely on electromagnetic acceleration of particles in the field of the strong wave itself.

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