Precise Test of SU(2) Chiral Breaking. A Coleman-Glashow Formula for Pseudoscalar Mesons

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As a precise and model-independent test of SU(2) chiral breaking ($\epsilon_3 u_3$), the analog of the Coleman-Glashow formula for pseudoscalar mesons is derived, and it is found to be satisfied within 1%. The presence of the term $\epsilon_3 u_3$ in the Hamiltonian density is therefore strongly supported by the experimental data.

In contrast with SU(3) breaking, SU(3)⊗SU(3) breaking[1] does not preserve the structure of the multiplets. The theoretical explanation of such a difference is that the vacuum is approximately SU(3) invariant, so that the physical states may still be classified according to SU(3) multiplets (at least to a first approximation), whereas the vacuum is not even approximately invariant under SU(3)⊗SU(3).[2] The nontrivial role of this symmetry is that of (approximate) automorphisms of the algebra of fields, with no simple transformation properties of the vacuum and particle states (spontaneously broken symmetry). This implies that one cannot use the Wigner-Eckart theorem to find relations between masses, coupling constants etc., as in the SU(2) or the SU(3) case, and it is more difficult to find clear and model-independent tests of the chiral-symmetry scheme.[1,2] Remedies to this difficulty have been the use of explicit Lagrangian models (like the σ model), perturbation expansions around the symmetry limit, approximate SU(3) invariance, asymptotic symmetry, questionable assumptions about the transformation properties of the vacuum (in order to be able to use the Wigner-Eckart theorem), etc. As a result one does not have a clear test of the *group-theoretical structure alone of the SU(3) ⊗ SU(3) scheme* namely: (i) the simple transformation properties of the fields, (ii) the $(\underline{3},\underline{3}^*)\oplus(\underline{3}^*,\underline{3})$ linear breaking in the Hamiltonian density

$$[G^\alpha,\varphi_i(x,t)]_{\text{E.T.}}=g_{ij}{}^\alpha\varphi_j(x,t), \tag{1}$$

$$H=H_0+\epsilon_0 u_0+\epsilon_8 u_8+\epsilon_3 u_3. \tag{2}$$

The above Hamiltonian is a generalization of the original Gell-Mann, Oakes, and Renner (GOR)[1] Hamiltonian since it allows ϵ_3 to be different from zero. The possible appearance of an explicit SU(2) breaking has been advocated several times in the past.[3] It has recently received much attention in connection with the determination of the Cabibbo angle.[4] However, apart from the possibility of explaining the Cabibbo angle, there does not seem to be strong and clear evidence in favor of this term. The difficulties connected with spontaneously broken symmetries, arising from the inapplicability of the Wigner-Eckart theorem, have been the main obstacle in fully exploiting the consequences of the Hamiltonian (2). The model-dependent discussions given in the literature are often in disagreement with one another and cannot be regarded as conclusive about the experimental support of the theory. The situation looks in fact rather open, and there is not a complete agreement about the value of ϵ_3.

The main purpose of this note is to discuss model-independent evidence of SU(2) chiral breaking ($\epsilon_3 u_3$). Our emphasis will be on the derivation of sum rules which follow only from the group-theoretical structure of the theory: In this way we will get the analog of the Coleman-Glashow formula for mesons, in extremely good agreement with the experimental data.

The striking success of the Coleman-Glashow formula for the baryon electromagnetic mass differences[5] does not have a counterpart in the 0^- meson case because of two main difficulties:

(a) In the baryon case the Coleman and Glashow argument gives two relations involving $M_{\Sigma\Lambda}$, which may therefore be eliminated to yield the Coleman-Glashow formula. In the 0^- meson case one has only one relation,

$$K^+ - K^0 + \pi^0 - \pi^+ = \sqrt{3}\, M_{\eta\pi}, \tag{3}$$

where $M_{\eta\pi}$ is the η-π matrix element of the Hamiltonian and K^0, π^0, etc. denote the squares of the particle masses. Since SU(3) alone does not provide a model-independent calculation of $M_{\eta\pi}$ (apart from the above equation), Eq. (3) cannot be tested.

(b) Independent information on $M_{\eta\pi}$ can be obtained by using the SU(3)$\otimes$SU(3) symmetry. In this case, however, if one puts $\epsilon_3 = 0$, one is faced with the Dashen paradox[6]: $K^+ - K^0 = \pi^+ - \pi^0$.

This indicates that *the meson electromagnetic mass differences cannot be explained by the standard electromagnetic Hamiltonian, and that in any case the strict analog of the Coleman-Glashow formula does not work in this case.*

The natural way out of the above difficulties is the introduction of the $\epsilon_3 u_3$ term in the Hamiltonian. In this way one may compute $M_{\eta\pi}$:

$$\sqrt{3}\, M_{\eta\pi} = \sqrt{2}\,\epsilon_3/F_\pi \simeq -7.9\times 10^{-3}\ (\mathrm{GeV}/c^2)^2. \tag{4}$$

The agreement with Eq. (3) is, however, not good: $K^+ - K^0 + \pi^0 - \pi^+ \simeq -5.2\times 10^{-3}$ $(\mathrm{GeV}/c^2)^2$, with a discrepancy of more than 30%.[7] The reason is that for pseudoscalar mesons the noninvariance of the vacuum plays a very important role in the breaking of SU(3)$\otimes$SU(3), and one cannot use the Wigner-Eckart theorem, on which both Eqs. (3) and (4) are based. As a matter of fact the SU(3) breaking due to the vacuum noninvariance is not negligible with respect to the explicit SU(3) breaking in the Hamiltonian[8]: $(\lambda_8/\lambda_0)/(\epsilon_8/\epsilon_0) \simeq 20\%$ $(\lambda_i \equiv \langle 0|u_i|0\rangle)$. This shows that SU(3) results based on the naive application of the Wigner-Eckart theorem might get reasonable corrections as in the case of the Gell-Mann–Okubo formula. The noninvariance of the vacuum was in fact crucial for the η-η' mixing and the identification of the η' particle.[8] A similar effect is therefore expected in the case of SU(2) chiral breaking since one has (again!)[8] $(\lambda_3/\lambda_0)/(\epsilon_3/\epsilon_0) \simeq 18$–$20\%$. *Results which are of order ϵ_3/ϵ_0 may therefore get corrections of the order 18–20% from the SU(2) noninvariance of the vacuum.* It is just in these cases that the Wigner-Eckart theorem is no longer useful, as discussed in the introduction, and one needs a different method. The Ward identity technique *à la* Glashow and Weinberg yields the right result.[8]

One gets in fact the following sum rules[9]:

$$\frac{F_K}{F_\pi}(K^+ - K^0)_{\epsilon_3} + \frac{F_\pi}{F_K}(\pi^0 - \pi^+)_{\epsilon_3} + \frac{\sqrt{2}\,\lambda_3 K_{\mathrm{av}}}{F_\pi} = \sqrt{3}\, M_{83}\left[\frac{1}{3}\left(4\frac{F_K}{F_\pi} - 1\right) - \frac{\sqrt{2}\,\lambda_3}{3F_K}\right] + \left(\tfrac{2}{3}\right)^{1/2} M_{03}\left[2\left(1 - \frac{F_K}{F_\pi}\right) - \frac{\sqrt{2}\,\lambda_3}{F_K}\right]$$
$$+ \frac{\sqrt{2}\,\lambda_3}{F_\pi} M_{33} + (\pi^0 - M_{33})\frac{F_\pi}{F_K}, \tag{5}$$

$$\sqrt{3}\, M_{83} = \frac{\sqrt{2}\,\epsilon_3}{F_\pi} - \frac{\sqrt{2}\,\lambda_3}{F_\pi}(M_{88} + \sqrt{2}\, M_{08}) = \frac{\sqrt{2}\,\epsilon_3}{F_\pi} - \frac{\sqrt{2}\,\epsilon_3}{F_\pi}[\eta + \sqrt{3}\,(\eta - \eta')\sin\theta\cos(\theta + \bar\theta)] + O\left(10^{-3}\frac{\sqrt{2}\,\epsilon_3}{F_\pi}\right), \tag{6}$$

$$\left(\tfrac{3}{2}\right)^{1/2} M_{03} = \frac{\sqrt{2}\,\epsilon_3}{F_\pi} - \frac{\sqrt{2}\,\lambda_3}{F_\pi}[\eta' + \left(\tfrac{3}{2}\right)^{1/2}(\eta - \eta')\sin\theta\sin(\theta + \bar\theta)] + O\left(10^{-3}\frac{\sqrt{2}\,\epsilon_3}{F_\pi}\right), \tag{7}$$

$$\frac{\sqrt{2}\,\lambda_3}{F_\pi} \simeq \frac{1}{K_{\mathrm{av}}}\left(\frac{\sqrt{2}\,\epsilon_3}{F_\pi} + \frac{F_K}{F_\pi}(K^0 - K^+) - \frac{F_\pi}{F_K}(\pi^0 - \pi^+)\right), \tag{8}$$

where M_{ij} is the mass matrix of the pseudoscalar mesons, $(K^+ - K^0)_{\epsilon_3}$ and $(\pi^0 - \pi^+)_{\epsilon_3}$ denote the contributions to the mass difference due to the term $\epsilon_3 u_3$, $K_{\mathrm{av}} \equiv K_0 - \frac{1}{2}(K^+ - K^0)_{\epsilon_3}$. It is not difficult to recognize in Eqs. (5) and (6) the analog of Eqs. (3) and (4) with the corrections arising from SU(3) noninvariance of the vacuum ($F_K \neq F_\pi$) and the SU(2) noninvariance of the vacuum ($\lambda_3 \neq 0$). In the limit of $F_K = F_\pi$, $\lambda_3 = 0$, Eqs. (5) and (6) reduce in fact to Eqs. (3) and (4).

By using a generalization of the Dashen theorem which takes into account the SU(3) noninvariance of the vacuum,[10] one may obtain an equation equivalent to Eq. (5), but involving only the observed mass differences ΔK, $\Delta\pi$:

$$\frac{F_K}{F_\pi}(K^+ - K^0) + \frac{F_\pi}{F_K}(\pi^0 - \pi^+) + \frac{\sqrt{2}\,\lambda_3}{F_\pi}K_{\mathrm{av}} = \text{right-hand side of Eq. (5)}. \tag{9}$$

The agreement with the experimental data is extremely good.[11] In $(\mathrm{GeV}/c^2)^2$ units,

$$\text{right-hand side of Eq. (9)} = \begin{cases} -7.77\times10^{-3} \text{ for } F_K/F_\pi = 1.22, \\ -8.09\times10^{-3} \text{ for } F_K/F_\pi = 1.28, \end{cases}$$

$$\text{left-hand side of Eq. (9)} = \begin{cases} -7.88\times10^{-3} \text{ for } F_K/F_\pi = 1.22, \\ -8.15\times10^{-3} \text{ for } F_K/F_\pi = 1.28. \end{cases}$$

Within a very good approximation the above formula can also be written in the simpler form

$$\frac{F_K}{F_\pi}(K^+ - K^0) + \frac{F_\pi}{F_K}(\pi^0 - \pi^+) + \frac{\sqrt{2}\,\lambda_3}{F_\pi} K_{\rm av} = \sqrt{3}\, M_{83}\left[\frac{1}{3}\left(\frac{4F_K}{F_\pi} - 1\right)\right]. \tag{10}$$

The agreement is still very good. It is worthwhile to remark that a precision of 1% in formula (10) is the best one can hope for since at this level of precision weak interactions or higher-order electromagnetic effects may play a non-negligible rôle.

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[7]To estimate $M_{\eta\pi}$ we have used the value of ϵ_3 obtained from an analysis based on the Ward identities *à la* Glashow and Weinberg [G. Parisi and M. Testa, Nuovo Cimento 67A, 13 (1970); G. Cicogna, F. Strocchi, and R. Vergara-Caffarelli, Phys. Rev. D 6, 301 (1972)]: $\epsilon_3 \simeq -0.3 m_\pi^3$, $\epsilon_0 \simeq 9.9 m_\pi^3$. This is close to the value of ϵ_3 suggested by Cabibbo and Maiani (Ref. 4). The other current values of ϵ_3 discussed in the literature, $\epsilon_3/\epsilon_8 \simeq 0.88/137$ (Gatto, Sartori, and Tonin, Ref. 4) $\epsilon_3/\epsilon_8 \simeq 0.05$ [R. J. Oakes, Phys. Lett. 29B, 683 (1969)], would lead to more violent disagreement.

[8]G. Cicogna, F. Strocchi, and R. Vergara-Caffarelli, Phys. Rev. D 6, 301 (1972).

[9]Equations (5)–(7) follow easily from the Ward identities $M_{ij}[G^\alpha,\lambda]_j = [G^\alpha,\epsilon]_i$ (for notations, see Ref. 8) by taking $i=3$ and $G^\alpha = (\sqrt{6}/F_\pi)F_8^5 - (\sqrt{2}/F_K)F_3^5$; $i=8$ and $G^\alpha = F_3^5$; $i=0$ and $G^\alpha = F_3^5$, respectively. Equation (8) is obtained from the fifth and sixth of Eqs. (65) of Ref. 8. [By the same Eqs. (65) one gets $M_{33} \simeq \pi^0 + M_{83}^2/(\eta - \pi)$.] It is important to stress that the Ward identities leading to Eqs. (5)–(8) do not involve the propagators of scalar fields and therefore the validity of Eqs. (5)–(6) does not require any assumption about the existence of the scalar partners of the nine pseudoscalar mesons (in particular the κ meson).

[10]The corrected form of Dashen theorem reads $(\Delta K)_{em} F_K/F_\pi = (\Delta\pi)_{em} F_\pi/F_K$. A more detailed discussion will be presented in a subsequent paper.

[11]The only unknown parameter in Eqs. (5)–(8) is ϵ_3. A possible way to determine ϵ_3 is to use the Ward identity

$$\frac{\sqrt{2}\,\epsilon_3}{F_\pi}\left(1 - \frac{K_{\rm av}}{\pi_s}\right) = -\frac{F_K}{F_\pi}(K^0 - K^+) + \frac{F_\pi}{F_K}(\pi^0 - \pi^+),$$

where

$$1/\pi_s \equiv \lim_{p\to 0} i\int e^{-ipx}\langle T[u_3(x)u_3(0)]\rangle\, dx.$$

Identifying π_s with the mass square of the $\delta(962)$ meson leads to the value of ϵ_3 obtained previously (Ref. 8): $\sqrt{2}\,\epsilon_3/F_\pi \simeq -8.0\times10^{-3}\ (\mathrm{GeV}/c^2)^2$. This value of ϵ_3 corresponds to a value of λ_3 consistent with the tadpole analysis by Coleman and Glashow (Ref. 3). It is important to remark that the check of Eq. (9) does not require a precise determination of ϵ_3. The accuracy of Eq. (9) is practically unaffected by reasonable variations of ϵ_3.