the S-wave two-nucleon forces, it is now very clear that accurate and extensive polarization data have provided the tests that prove the necessity for the inclusion of higher partial waves in the calculations.

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[1]W. Haeberli, in *Three-Body Problem in Nuclear and Particle Physics*, edited by J. S. C. McKee and P. M. Rolph (North-Holland, Amsterdam, 1970), p. 188, and references therein.

[2]R. Aaron, R. D. Amado, and Y. Y. Yam, Phys. Rev. 140, B1291 (1965).

[3]I. H. Sloan, Phys. Rev. 185, 1361 (1969), and Nucl. Phys. A168, 211 (1971).

[4]L. D. Faddeev, Zh. Eksp. Teor. Fiz. 39, 1459 (1960) [Sov. Phys. JETP 12, 1014 (1961)]; C. Lovelace, Phys. Rev. 135, B1225 (1964).

[5]S. C. Pieper and K. L. Kowalski, Phys. Rev. C 5, 306 (1972).

[6]J. C. Aarons and I. H. Sloan, Nucl. Phys. A182, 369 (1972).

[7]P. Doleschall, Phys. Lett. 38B, 298 (1972).

[8]P. Doleschall, Phys. Lett. 40B, 443 (1972).

[9]S. C. Pieper, Nucl. Phys. A193, 529 (1972), and in Proceedings of the International Conference on Few Particle Problems in the Nuclear Interaction, Los Angeles, August 1972 (North-Holland, Amsterdam, to be published), and private communication.

[10]I. H. Sloan, Phys. Rev. 165, 1587 (1968), and 185, 1361 (1969).

[11]P. Doleschall, J. C. Aarons, and I. H. Sloan, Phys. Lett. 40B, 605 (1972).

[12]J. Arvieux, R. Beurtey, J. Goudergues, B. Mayer, A. Papineau, and J. Thirion, Nucl. Phys. A102, 503 (1967).

[13]G. R. Satchler, Nucl. Phys. 8, 65 (1958).

[14]Ch. Leemann, H. E. Conzett, W. Dahme, J. MacDonald, and J. P. Meulders, Bull. Amer. Phys. Soc. 17, 562 (1972), and to be published.

[15]In a paper contributed to the International Conference on Few Particle Problems in the Nuclear Interaction, Los Angeles, August 1972 (North-Holland, Amsterdam, to be published), A. Fiore, J. Arvieux, N. Van Sen, G. Perrin, F. Merchez, J. C. Gondrand, C. Perrin, J. L. Durand, and R. Draves-Blanc report measurements of vector analyzing powers in d-p scattering at 20 and 30 MeV which are in good agreement with our results.

[16]K. L. Kowalski and S. C. Pieper, Phys. Rev. C 5, 324 (1972).

[17]Y. Yamaguchi and Y. Yamaguchi, Phys. Rev. 95, 1635 (1954); Y. Yamaguchi, Phys. Rev. 95, 1628 (1954).

Perturbations on the Mixmaster Universe*

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Mathematical formalisms for the separation and solution of the tensor perturbation equations in an empty, diagonal, type-IX space is developed, based upon group-symmetry properties of homogeneous spaces. Numerical results in sampling solutions of the "mixmaster universe" show damping amplitudes of perturbations as the universe expands, a behavior in qualitative accordance with earlier results on the Friedmann universe.

In an attempt towards the construction of the general cosmological solution to the Einstein equation, Lifshitz and Khalatnikov have shown that near the singularity, solutions containing matter manifest no features not already found in the vacuum solutions.[1,2] Later they discovered that the asymptotic behavior of the metric near the singularity is at each point of the singular hypersurface described by a mixmaster-type behavior.[3] This discovery has put the mixmaster universes[4,5] among the most viable models for studying the general cosmological solution. To probe deeper into the earlier states of the universe near the singularity, it is important to understand the behavior of perturbations in the mixmaster universe: Given an anisotropic metric with small inhomogeneities, how would they behave as the universe evolves? Will inhomogeneous perturbations damp away and the Khalatnikov-Lifshitz generic case be restored, or will they grow to such an extent as to disrupt the stability of the universe? And if so, how? It was towards an understanding of these problems that the present work was first motivated.

We have studied tensor perturbations to an empty, diagonal, mixmaster universe.[4] In this Letter we shall work out a mathematical formalism for the separation and solution of the perturbation equations. Unlike earlier studies on the perturbation problem, such as the Schwarzschild metric[6] or the Friedmann universe,[1] in which complete sets of basis tensor harmonics have to be constructed explicitly for the expansion of tensor perturbations, the method presented here makes use of purely group-symmetry properties of homogeneous spaces[7] and is readily generalized to other types of homogeneous cosmologies. The method presented here has two particular features:

(1) Since every point in a homogeneous space is equivalent to any other point by a group operation, then instead of trying to construct basis tensor harmonics (such as the tensor spherical and hyperspherical harmonics for the Schwarzschild and Friedmann solutions, respectively) as functions of the whole space, we can *choose* to evaluate the perturbation equations at one arbitrary point of space and regenerate the complete solutions by means of group translations on the manifold. In the present exposition we use for the invariant basis a coordinate representation in Euclidean four-dimensional space with restrictions on the three-sphere. Here, a convenient point of choice that will reduce the problem to maximum simplicity is, without doubt, the pole (where $x_1=x_2=x_3=0$, $x_4=1$; x_i are the coordinates in E^4).

(2) If we choose to evaluate the equations at the pole, since for our purpose here we need only make use of the second derivatives of the metric tensor (from which the curvature tensors are readily computable), the expansion of the metric tensor components in powers of x_i can be terminated at the second order. This is most easy to carry out and simplifies the problem considerably. The coefficients of expansion of the linear term give the Christoffel symbols, that of the quadratic term gives components of the curvature tensor.[8]

We outline briefly our approach in the following.

The metric of a homogeneous cosmology is in general given by

$$ds^2=-dt^2+\bar{\gamma}_{ab}(t)\sigma^a\sigma^b \quad (a,b=1,2,3),$$

where the σ^a are the invariant differential forms of the space. (Throughout this paper, summations are extended over repeated indices, unless otherwise stated.) They obey the exterior differential relations $d\sigma^a=\frac{1}{2}C_{bc}{}^a\sigma^b\wedge\sigma^c$, where $C_{bc}{}^a$ is the structure constants of the underlying symmetry group and the caret denotes the "wedge" product. For the mixmaster space, which belongs to the Bianchi type-IX classification, the group is SO(3) and the $C_{bc}{}^a$ is equal to ϵ_{abc}, the antisymmetric tensor. Expressed in terms of the coordinate differentials dx^i of the four-dimensional Euclidean space E^4, the invariant basis forms σ^a on the three-sphere S^3 are given by[9]

$$\begin{aligned}
\sigma^1&=2(-x_4dx_1-x_3dx_2+x_2dx_3+x_1dx_4),\\
\sigma^2&=2(x_3dx_1-x_4dx_2-x_1dx_3+x_2dx_4),\\
\sigma^3&=2(-x_2dx_1+x_1dx_2-x_4dx_3+x_3dx_4),\\
\sigma^4&=2(x_1dx_1+x_2dx_2+x_3dx_3+x_4dx_4).
\end{aligned}\tag{1}$$

Introducing the transformation matrices $S_{ai}(x)$ defined by $\sigma^a=2S_{ai}(x)dx^i$, the spatial metric can be written as

$$dl^2=\gamma_{ab}(t)S_{ai}(x)S_{bj}(x)dx^i dx^j=g_{ij}(x,t)dx^i dx^j,\tag{2}$$

where now $g_{ij}(x,t)=\gamma_{ab}(t)S_{ai}(x)S_{bj}(x)$ and $\gamma_{ab}=4\bar{\gamma}_{ab}$. The diagonal mixmaster metric takes the form $\gamma_{ab}(t)=l_a{}^2(t)\delta_{ab}$, where l_a $(a=1,2,3)$ are the three principal axes of the universe.

For subsequent calculations, we need explicit expressions for the metric tensor $g_{ij}(x,t)$. Doing this straightforwardly would involve lengthy algebraic manipulations. However, the calculation can be greatly simplified if we take note of the above-stated properties of a homogeneous space. That is, we can evaluate all geometric field quantities at any arbitrary point in space; and at the pole it suffices to retain up to the quadratic terms in x_i in the series expansion of $g_{ij}(x,t)$. Hence, from (1) and (2), writing

$$x_4^2 = 1 - \sum_{i=1}^{3} x_i^2, \quad dx_4 = -\sum_{i=1}^{3} x_i\,dx_i\,,$$

also setting $x_4 = 1$, we deduce the following algebraic relation for the general type-IX metric in E^4 coordinates on the three-sphere:

$$g_{ij}(x,t) = \gamma_{ij}(t) + \epsilon_{kjl}\gamma_{ik}x_l + \epsilon_{kil}\gamma_{kj}x_l + (\gamma_{mm})x_i x_j + 2\gamma_{kl}\epsilon_{kim}\epsilon_{ljn}x_m x_n + [(\gamma_{mn}x_m x_n) - (\gamma_{mn})(x_n x_n)]\,\delta_{ij}. \tag{3}$$

From this expansion, the Christoffel symbols and their derivatives at the pole are readily computed. All the nonzero components are given as follows (no summation over repeated indices):

$$\Gamma_{ij}{}^{0} = \kappa_i \gamma_i \delta_{ij}, \quad \Gamma_{0j}{}^{i} = \kappa_i \delta_{ij}, \quad \Gamma_{jk}{}^{i} = \epsilon_{ijk}(\gamma_k - \gamma_j)/\gamma_i, \quad \partial\Gamma_{ij}{}^{0}/\partial x_k = \epsilon_{ijk}(\kappa_i/\gamma_i - \kappa_j/\gamma_j),$$
$$\partial\Gamma_{0j}{}^{i}/\partial x_k = \epsilon_{ijk}(\kappa_i - \kappa_j), \quad \partial\Gamma_{jk}{}^{i}/\partial x_l = \begin{cases} 1 \text{ for } i=j=k=l, \\ (\gamma_p - \gamma_i)(\gamma_p + \gamma_i - \gamma_k)/\gamma_p\gamma_i \text{ for } i=j\neq k=l \quad (p\neq i\neq k), \\ (\gamma_i + 2\gamma_j - 2\gamma_p)/\gamma_i \text{ for } j=k\neq i=l, \end{cases} \tag{4}$$

where $\gamma_i \equiv l_i^2$, $\kappa_i \equiv \dot{l}_i/l_i$.

Perturbations to a spatially homogeneous metric can, in general, be expressed in terms of the basis-invariant forms of the space with the time-dependent expansion coefficients coupling to the representation functions of the particular underlying symmetry group of the space. For type-IX spaces, the representation functions are the well-known Wigner D functions $D_{KM}{}^{J}(g)$. The general perturbation $h_{ij}{}^{JM}(x,t)$ belonging to definite angular-momentum states (J,M) can be written as

$$h_{ij}{}^{JM}(x,t) = \sum_{K=-J}^{J} h_{ab}{}^{K}(t)\sigma_{(i)}{}^{a}\sigma_{(j)}{}^{b}D_{KM}{}^{J}(g) = \sum_{K} h_{ij}{}^{K}(x,t)D_{KM}{}^{J}(g), \tag{5}$$

where g is a group element of SO(3) and $h_{ij}{}^{K}(x,t) = 4h_{ab}{}^{K}(t)S_{ai}(x)S_{bj}(x)$. The time-dependent amplitude functions $h_{ab}{}^{K}(t)$ are governed by a set of coupled differential equations, to be derived from the perturbation equations below. For each definite value of J, there exist $2J+1$ components of $h_{ij}{}^{K}(x,t)$ coupled to the Wigner functions in the intrinsic magnetic quantum number K. No such coupling exists for spaces of higher symmetry, such as the Taub or the Friedmann universe. The diagonal mixmaster metric possesses a further symmetry under the four-group (invariance under rotation through π about any of the principal curvature axes), and the eigenfunctions are the symmetrized symmetric-top wave functions. Detailed expositions on the solutions of scalar wave equations and the symmetry classifications of the wave functions in the mixmaster universe have been given by Hu.[10]

Now with the Christoffel symbols and their derivatives given by (4), we can proceed to simplify the perturbation equations for the mixmaster universe by evaluating them at the pole $(x_1 = x_2 = x_3 = 0)$. The perturbation equations on an empty background metric (see, e.g., Ref. 6)

$$2\delta R_{\mu\nu} = h_{\mu\nu;\alpha}{}^{;\alpha} - h_{\mu\alpha;\nu}{}^{;\alpha} - h_{\nu\alpha;\mu}{}^{;\alpha} + h^{\alpha}{}_{\alpha;\mu;\nu} \tag{6}$$

are first expressed in terms of ordinary derivatives. (Here Greek indices range from 0 to 3, Latin indices from 1 to 3, and semicolons denote covariant derivatives with respect to the four-dimensional background metric.) For the perturbation tensor components $h_{\mu\nu}$, we can avail ourselves of the freedom in the choice of gauge to impose the synchronous conditions $h_{00} = h_{0i} = 0$. In simplifying the perturbation equations we shall need the first and second derivatives of $h_{ij}{}^{K}(x,t)$ and $D_{KM}{}^{J}(g)$. This can be done with relative ease if we make use of the group-symmetry properties of the space. The first derivative of $h_{ij}{}^{K}$ can be related to $h_{ij}{}^{K}$ themselves by means of the Killing conditions. Alternatively, the spatial derivatives of $h_{ij}{}^{K}$ evaluated at the pole can be obtained by making an expansion of $h_{ij}{}^{K} = h_{ab}{}^{K}(t)\sigma^a\sigma^b$ in powers of x_i in exactly the same way as was done for the metric tensor. In fact, we can read off the relations from (3), with $h_{ij}{}^{K}$ replacing γ_{ij}. The spatial derivatives of the D functions can best be understood from the action of the invariant operators $\hat{e}_i$ (dual to the basis forms σ^a) which are simply related to the intrinsic angular-momentum operators $\hat{L}_i$ by $\hat{L}_i = i\hat{e}_i$ $(i = 1,2,3)$, where the $\hat{L}_i$ satisfy the commutation relations $[\hat{L}_1, \hat{L}_2] = -i\hat{L}_3$ (cyclic). From (1) it is easy to see that $\partial/\partial x_i$

$=2S_{ai}\hat{e}_a$. At the pole, $S_{ai}=-\delta_{ai}$ and hence

$$\partial/\partial x_i|_0 = -2\hat{e}_i = 2i\hat{L}_i\,. \tag{7a}$$

Repeated operations of $\hat{L}_i$ yield the following relations for the second spatial derivatives of $D_{KM}{}^J$ at the pole:

$$\frac{\partial^2}{\partial x_i{}^2}\bigg|_0 = -4\hat{L}_i{}^2,\quad \frac{\partial^2}{\partial x_i\partial x_j}\bigg|_0 = \frac{\partial^2}{\partial x_j\partial x_i}\bigg|_0 = -2(\hat{L}_i\hat{L}_j+\hat{L}_j\hat{L}_i)\quad (i,j=1,2,3). \tag{7b}$$

The operations of the angular-momentum operators on the representation functions are well known. Define the ladder operators $\hat{L}_\pm = \hat{L}_1 \pm i\hat{L}_2$. From the elementary relations

$$\hat{L}_+D_K = i\epsilon_K D_{K-1},\quad \hat{L}_-D_K = -i\epsilon_{K+1}D_{K+1},\quad \hat{L}_3D_K = KD_K, \tag{8}$$

where $\epsilon_K \equiv [(J+K)(J-K+1)]^{1/2}$, the action of $\hat{L}_i$ and $\hat{L}_i\hat{L}_j$ can easily be derived.

After a lengthy but straightforward calculation, we arrive at Eqs. (9) for the time-dependent expansion coefficients $h_{ab}{}^K(t)$. The other four equations are obtained from δR_{11}, δR_{12} by cyclically permuting the indices 1, 2, 3 on h_{ij}, γ_{ij} and ∂_i, ∂_j. Here the spatial derivatives ∂_i are understood to be acting on the D_K functions with the effect of shifting the K indices of the components. The final expressions after the operation of $\partial_i\partial_j$ are obtained via (7) and (8). Because of lack of space, we give in (10) only δR_{12} as an example:

$$\begin{aligned}
2\delta R_{11} = \Big\{ &-\ddot{h}_{11}{}^K + (3\kappa_1-\kappa_2-\kappa_3)\dot{h}_{11}{}^K - 4\kappa_1{}^2 h_{11}{}^K - \kappa_1\gamma_1\dot{h}^K + h_{11}{}^K\Delta - 2\sum_{n=1}^{3}\frac{h_{1n}{}^K}{\gamma_n}\partial_1\partial_n + h^K\partial_1{}^2 \\
&+ 4h_{23}{}^K\left(\frac{1}{\gamma_3}-\frac{1}{\gamma_2}\right)\partial_1 + 2h_{12}{}^K\left(-\frac{2}{\gamma_3}+\frac{1}{\gamma_2}-\frac{2\gamma_1}{\gamma_2\gamma_2}\right)\partial_3 + 2h_{13}{}^K\left(\frac{2}{\gamma_2}-\frac{1}{\gamma_3}+\frac{2\gamma_1}{\gamma_2\gamma_3}\right)\partial_2 \\
&+\frac{4}{\gamma_2\gamma_3}\left[-2\gamma_1{}^2\left(\frac{h_{11}{}^K}{\gamma_1}\right) + (\gamma_1{}^2+\gamma_2{}^2-\gamma_3{}^2)\left(\frac{h_{22}{}^K}{\gamma_2}\right) + (\gamma_1{}^2+\gamma_3{}^2-\gamma_2{}^2)\left(\frac{h_{33}{}^K}{\gamma_3}\right)\right]\Big\} D_K(0) = 0, \\
2\delta R_{12} = \Big\{ &-\ddot{h}_{12}{}^K + (\kappa_1+\kappa_2-\kappa_3)\dot{h}_{12}{}^K - 4\kappa_1\kappa_2 h_{12}{}^K + \frac{4h_{12}{}^K}{\gamma_1\gamma_2\gamma_3}[\gamma_3{}^2-(\gamma_1+\gamma_2)^2] \\
&+\frac{h_{12}{}^K}{\gamma_3}\partial_3{}^2 + \frac{h_{33}{}^K}{\gamma_3}\partial_1\partial_2 - \frac{h_{13}{}^K}{\gamma_3}(\partial_2\partial_3+3\partial_1) - \frac{h_{23}{}^K}{\gamma_3}(\partial_1\partial_3-3\partial_2) \\
&+\left[h_{11}{}^K\left(\frac{\gamma_2-\gamma_3+3\gamma_1}{\gamma_1\gamma_3}\right) + h_{22}{}^K\left(\frac{\gamma_3-\gamma_1-3\gamma_2}{\gamma_2\gamma_3}\right) + h_{33}{}^K\left(\frac{\gamma_2-\gamma_1}{\gamma_3{}^2}\right)\right]\partial_3\Big\} D_K(0) = 0, \\
2\delta R_{00} = \sum_{m=1}^{3}&\frac{1}{\gamma_m}(\ddot{h}_{mm}{}^K - 2\dot{\kappa}_m h_{mm}{}^K - 2\kappa_m\dot{h}_{mm}{}^K) = 0, \\
2\delta R_{0i} = \Big\{ &\sum_{m\neq i}\frac{1}{\gamma_m}\{[\dot{h}_{mm}{}^K - (\kappa_i+\kappa_m)h_{mm}{}^K]\partial_i - (\dot{h}_{im}{}^K - 2\kappa_i h_{im}{}^K)\partial m\} \\
&+2\epsilon_{ijk}\left[\left(\frac{1}{\gamma_k}-\frac{1}{\gamma_j}\right)\dot{h}_{jk}{}^K + 2\left(\frac{\kappa_k}{\gamma_j}-\frac{\kappa_j}{\gamma_k}\right)h_{jk}{}^K\right]\Big\} D_K(0) = 0 \text{ (no sum on } j,k);
\end{aligned} \tag{9}$$

$$\begin{aligned}
2\delta R_{12} = &-\ddot{h}_{12}{}^K + (\kappa_1+\kappa_2-\kappa_3)\dot{h}_{12}{}^K - 4\kappa_1\kappa_2 h_{12}{}^K + \frac{4h_{12}{}^K}{\gamma_1\gamma_2\gamma_3}[\gamma_3{}^2-(\gamma_1+\gamma_2)^2] - \frac{4h_{12}{}^K}{\gamma_3}K^2 \\
&- i\frac{h_{33}{}^{K+2}}{\gamma_3}\epsilon_{K+2}\epsilon_{K+1} + i\frac{h_{33}{}^{K-2}}{\gamma_3}\epsilon_{K-1}\epsilon_K + \frac{2(K+1)}{\gamma_3}(h_{13}{}^{K+1} + ih_{23}{}^{K+1})\epsilon_{K+1} + \frac{2(K-1)}{\gamma_3}(h_{13}{}^{K-1} - ih_{23}{}^{K-1})\epsilon_K \\
&+ 2iK\left[h_{11}{}^K\left(\frac{\gamma_2-\gamma_3+3\gamma_1}{\gamma_1\gamma_3}\right) + h_{22}{}^K\left(\frac{\gamma_3-\gamma_1-3\gamma_2}{\gamma_2\gamma_3}\right) + h_{33}{}^K\left(\frac{\gamma_2-\gamma_1}{\gamma_3{}^2}\right)\right] = 0,
\end{aligned} \tag{10}$$

where

$$\gamma_i \equiv l_i{}^2,\quad \kappa_i \equiv \dot{l}_i/l_i,\quad h^K \equiv \sum_{i=1}^{3} h_{ii}{}^K/\gamma_i\,,\quad \partial_i \equiv \frac{\partial}{\partial x_i}\bigg|_0,\quad \Delta \equiv \sum_{i=1}^{3}\frac{1}{\gamma_i}\frac{\partial^2}{\partial x_i{}^2},$$

and the dot denotes differentiation with respect to time t.

The final result consists of a set of six coupled second-order ordinary differential equations, δR_{ij}, $i,j=1,2,3$ (the dynamic equations) and four first-order ones δG_{00}, δR_{0i} (the constraint equations) for the six unknown functions $h_{ab}{}^K(t)$ $(a,b=1,2,3)$. They are checked to be consistent in time. For each J value, there are thus $6(2J+1)$ dynamic equations and $4(2J+1)$ constraint equations for $6(2J+1)$ unknowns $h_{ab}{}^K(t)$. Since the dynamic equations are second order in time, as initial conditions, we need to specify h_{ab} and $\dot{h}_{ab}$, the derivative with respect to time. Thus a total of $8(2J+1)$ initial conditions have to be specified; the other $4(2J+1)$ are solved from the constraint equations. Because of the consistency of the equations, the constraint equations are always satisfied.

We have solved the perturbation equations numerically on a computer in the simplest case of $J=\frac{1}{2}$ for several sampling runs of the mixmaster solutions. The behavior of the lowest mode is already sufficient for determining the stability of the universe. In all cases the perturbation amplitudes decrease as the volume of the universe increases (receding from singularity) and vice versa. The overall characteristic behavior encompasses the qualitative description of perturbations (gravitational waves) in the Friedmann universe (which is a special case of the mixmaster universe with l_i all equal) as given earlier by Lifshitz and Khalitnikov.[1] Their equations are derivable from the ones given here, only that the mixmaster does not yield separation of different types of perturbations—scalar, vector, and tensor, as is possible for the Friedmann case.

Now that the mathematical formalism is well laid out, studies on perturbations to a mixmaster universe containing matter and the related problems of galaxy formation, which constitute topics of even greater physical interest, can easily be extended. This, together with the details of the present work, are to be reported later elsewhere.

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[1]E. M. Lifshitz and I. M. Khalatnikov, Advan. Phys. 12, 185 (1963).

[2]E.g., for a discussion on the structure of a "velocity-dominated" singularity, cf. D. Eardley, E. Liang, and R. Sachs, J. Math. Phys. 13, 99 (1972).

[3]V. A. Belinskii, E. M. Lifshitz, and I. M. Khalatnikov, Advan. Phys. 19, 525 (1970).

[4]C. W. Misner, Phys. Rev. Lett. 22, 1071 (1969).

[5]I. M. Khalatnikov and E. M. Lifshitz, Phys. Rev. Lett. 24, 76 (1969).

[6]T. Regge and J. A. Wheeler, Phys. Rev. 108, 1063 (1957).

[7]The analogy of the method of solution to this problem with that of the quantum mechanics of the electron was earlier suggested by J. A. Wheeler.

[8]This method was originally due to B. Riemann, in *The Collected Works of Bernhard Riemann*, edited by H. Weber (Dover, New York, 1953), p. 272.

[9]See, e.g., C. W. Misner, J. Math. Phys. (N. Y.) 4, 924, Appendix B (1963).

[10]B. L. Hu, Ph.D. thesis, Princeton University, 1972 (unpublished); see also Bull. Amer. Phys. Soc. 17, 450 (1972).