from $J = 0$ expectation values using Racah algebra techniques. The splittings from H^6 and H_N' were calculated with a sequence of eight wave functions, the results with the largest wave function (165 terms) being given in Table I. For each wave function the total splittings from H^6 and H_N' were calculated and the four sequences of values extrapolated to give the entries of Table II. The convergence is quite smooth and the accuracy obtained is more than sufficient for the present comparison with experiment.

The separate contributions and totals are listed in Table III. ν_{01} agrees well with experiment and yields a value of α accurate to ± 3 ppm and consistent with Ref. 1. Considerable work remains to find ν_{12} to comparable accuracy (~ 10 ppm). Completion and improvement in accuracy of the second-order calculation is mandatory, and at least an estimate of the $\alpha^7 mc^2$ terms is desirable. Consistency of the two values will provide a nontrivial check of the computations. A complete and independent recalculation of the effects evaluated here would provide increased confidence. It does appear that the helium fine structure will ultimately provide a value of α accurate to better than one part per million.

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Higher-Order Vacuum Polarization Corrections in Muonic Atoms*

M. K. Sundaresan and P. J. S. Watson
Department of Physics, Carleton University, Ottawa, Canada
(Received 20 April 1972)

In view of the recently reported discrepancies between theoretical calculations and accurate experimental measurements of muonic x-ray energies for several elements in the energy range 150–440 keV, we have recalculated the vacuum polarization corrections. Our results reduce, but do not eliminate, the discrepancies. We discuss the effects of a possible anomalous coupling of the muon to a speculative scalar meson of the type suggested by Weinberg.

In a recent paper on the experimental test of the theory of muonic atoms, Dixit *et al.*[1] have reported measurements of muonic x-ray energies in a number of elements in the energy range 150–440 keV. They picked for measurement only the higher transitions in atoms with spherical nuclei and made special efforts to measure these transitions with an absolute precision of 15–21 eV. Since these high transitions are only a little affected by the nucleus, the measurements could serve to test the extent to which the existing theory of such hydrogenlike atoms, namely that based on the Dirac equation together with applicable atomic and vacuum-polarization corrections, works. The prinicpal correction to the Dirac energy in this energy range turns out to be that due to e^--e^+ vacuum polarization. These corrections were calculated by Fricke[2]; and, using these calculated numbers, Dixit *et al.*[1] arrived at their theoretical values for the transition energies. Corrections due to nuclear polarization, the screening effect due to electrons, the Lamb shift and relativistic corrections to the reduced mass, etc. are small but were nevertheless taken into

account in arriving at the theoretical values. Dixit *et al.* concluded from their work that the theoretical values for the transition energies, after taking all known corrections to the Dirac energy into account, were consistently larger than the experimentally measured values. As already mentioned, since the principal correction is that due to vacuum polarization, the question arose as to whether these corrections are being calculated correctly. In view of the importance of this question and the implications of the discrepancy between theory and experiment, we decided to recalculate the vacuum-polarization corrections.

This calculation was done in a straightforward manner. As is well known, the contribution to the vacuum polarization from e^--e^+ pairs is made up of the following parts: (a) the Uehling term, of order α, (b) terms of order α^2, and (c) terms of higher order in $Z\alpha$. Of these, the contributions due to (a) and (b) increase the binding energy of the muon, whereas the contribution from (c), for example, the term of order $(Z\alpha)^3$, *decreases* the binding energy of the muon. Fricke,[2] in his work, does not have the correct sign for the $(Z\alpha)^3$ contribution. We have taken the well-known expressions for the modification of the photon propagator given to any order in α in momentum space and directly calculated the energy shift of the muon level from the formula

$$\Delta E = \frac{2Z\alpha}{\pi}\int_0^\infty dq\, q^2 \times\left[\int_0^\infty (g^2+f^2)r^2\, dr \frac{\sin qr}{qr}\right]\frac{1}{q^2}\,U(q); \quad (1)$$

$\alpha = 137.036\,02$ is the fine structure constant, g and f are the respective large and small components of the Dirac wave function for the muon, and $U(q)$ represents the propagator modification with q the three-dimensional momentum transfer. The muon mass in this work has been taken to be 105.6599 MeV. For the Uehling term, $U(q)$ is

$$\frac{2\alpha}{\pi}\int_0^1 dz\, z(1-z)\ln\left[1+\frac{q^2}{m_e^2}z(1-z)\right], \quad (2)$$

where m_e is the electron mass, 0.511 004 1 MeV. For the terms of order α^2, four Feynman diagrams contribute as shown in Fig. 1. The first of these involves simply the square of $U(q)$ for the Uehling term above. A combined expression for the modification due to diagrams (b), (c), and (d) of Fig. 1 in a form suitable for numerical computations has been given by Kallen and Sabry,[3]

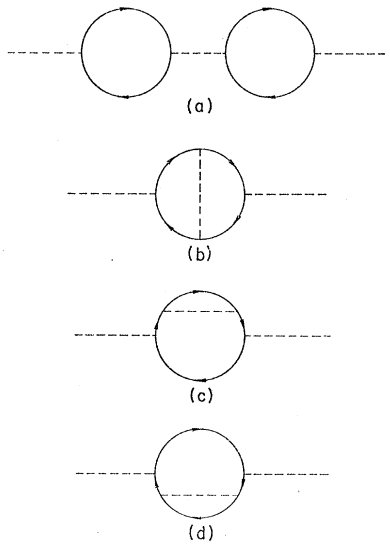

FIG. 1. Feynman diagrams for α^2 corrections.

quoted by Fricke[2] in his paper in Eqs. (18) and (19). One misprint has been found in this expression: The sign of the term $\frac{1}{12}\delta^6\ln^2(1+\delta/|1-\delta|)$ is incorrect. Using these expressions in Eq. (1), the contribution of order α^2 was obtained by carrying out the q integral numerically with a twenty-point Gaussian quadrature. We checked our results for the shifts in uranium ($Z=92$) with those published by Fricke.[2] Our results (in keV) for the combined effect of diagrams (b)–(d) are, $3d_{5/2}$, 0.0956; $4f_{7/2}$, 0.0353; $5g_{9/2}$, 0.0155; $3d_{3/2}$, 0.1051; $4f_{5/2}$, 0.0366; and $5g_{9/2}$, 0.0158; which are to be compared with Fricke's numbers, for $3d_{3/2}$, 0.099; $3d_{5/2}$, 0.095; and $4f_{5/2}$, 0.033; which are in reasonable agreement. For the contribution from diagram (a) in Fig. 1 for uranium we get, for $3d_{5/2}$, 0.0386; $4f_{7/2}$, 0.0102; $5g_{9/2}$ 0.0030; $3d_{3/2}$, 0.0437; $4f_{5/2}$, 0.0108; and $5g_{7/2}$, 0.0031; which are to be compared with Fricke's values, for $3d_{3/2}$, 0.060; $3d_{5/2}$, 0.060; and $4f_{5/2}$, 0.020. For these the agreement is not so good.

For calculating the higher-order Z corrections, we used the calculations of Wichmann and Kroll[4] to generate the $U(q)$ needed in Eq. (1) for obtaining the energy shifts. Fricke has also given results for these contributions, and has given the same sign as for the α^2 corrections. This is erroneous; the sign of these contributions should be opposite to that of the contribution from the Uehling term as is apparent from Ref. 4. Wichmann and Kroll have given expressions for a quantity called $U(p)$ in Eq. (58) of their paper, which is closely related to the $U(q)$ we want. The

TABLE I. Calculated energy shifts for α^2 and higher $Z\alpha$ corrections.

Z	Level	ΔE(kev) α^2 Fig.1 (a)	ΔE(kev) α^2 Fig.1 (b,c,d)	ΔE(kev) $(Z\alpha)^3$ and higher
47	$3d_{5/2}$	0.0043	0.0154	-0.0166
	$3d_{3/2}$	0.0045	0.0158	-0.0169
	$4f_{7/2}$	0.0008	0.0052	-0.0072
	$4f_{5/2}$	0.0008	0.0053	-0.0072
48	$3d_{5/2}$	0.0046	0.0163	-0.0182
	$3d_{3/2}$	0.0048	0.0168	-0.0185
	$4f_{7/2}$	0.0009	0.0056	-0.0079
	$4f_{5/2}$	0.0009	0.0056	-0.0080
50	$3d_{5/2}$	0.0053	0.0183	-0.0218
	$3d_{3/2}$	0.0055	0.0188	-0.0223
	$4f_{7/2}$	0.0010	0.0063	-0.0095
	$4f_{5/2}$	0.0011	0.0063	-0.0096
56	$3d_{5/2}$	0.0078	0.0249	-0.0361
	$3d_{3/2}$	0.0082	0.0258	-0.0370
	$4f_{7/2}$	0.0017	0.0087	-0.0161
	$4f_{5/2}$	0.0017	0.0088	-0.0163
	$5g_{9/2}$	0.0003	0.0036	-0.0079
	$5g_{7/2}$	0.0004	0.0036	-0.0081
82	$4f_{7/2}$	0.0068	0.0256	-0.0930
	$4f_{5/2}$	0.0071	0.0264	-0.0950
	$5g_{9/2}$	0.0019	0.0111	-0.0485
	$5g_{7/2}$	0.0020	0.0113	-0.0490

precise relation is

$$U(q) = (2iq)^{-1}\int_{-iq}^{+iq} Q(p')\,dp', \tag{3}$$

where $Q(p)$ is the Laplace transform of the induced charge density for which limiting forms are given for small and large p. We have found a continued-fraction interpolation in $1/p$, which interpolates for $Q(p)$ smoothly between the large p value given in Eqs. (69) and (70) of Wichmann and Kroll's paper and Eq. (59) of the same paper, valid for small p. The instability and accuracy of the continued-fraction interpolation was checked by varying the input points, and insignificant changes were produced in the numerical results. This interpolated expression was used to obtain $U(q)$ from Eq. (3), which was then used to find the the shifts from Eq. (1). The effect of the change in the sign of the higher-order $Z\alpha$ contributions is that the combined effect of the α^2 and $Z\alpha$ terms is much reduced. For high-Z elements, the higher $Z\alpha$ contributions win out over the α^2 contributions, thus giving a net decrease in the binding energy of the muon for high-Z elements.

We give below the results of our calculations in Tables I and II. In Table I we give the shifts for the α^2 and higher $Z\alpha$ terms for the levels of interest. Taking the values for the shifts as given in Table I, and taking all other corrections as given by Dixit *et al.*, we give in Table II (column

TABLE II. Comparison of corrected transition energies with experiment.

Z	Transition	E_{th}(kev)	E(expt)(kev)	E'_{th}(kev)
47	$4f_{5/2} - 3d_{3/2}$	308.464	308.428 ± 0.019	308.422
"	$4f_{7/2} - 3d_{5/2}$	304.781	304.759 ± 0.017	304.741
48	$4f_{5/2} - 3d_{3/2}$	321.997	321.973 ± 0.018	321.951
"	$4f_{7/2} - 3d_{5/2}$	317.990	317.977 ± 0.017	317.946
50	$4f_{5/2} - 3d_{3/2}$	349.982	349.953 ± 0.020	349.929
"	$4f_{7/2} - 3d_{5/2}$	345.257	345.226 ± 0.018	345.206
56	$4f_{5/2} - 3d_{3/2}$	441.367	441.299 ± 0.021	441.288
"	$4f_{7/2} - 3d_{5/2}$	433.912	433.829 ± 0.019	433.837
"	$5g_{7/2} - 4f_{5/2}$	201.280	201.260 ± 0.016	201.265
"	$5g_{9/2} - 4f_{7/2}$	199.913	199.902 ± 0.015	199.898
82	$5g_{7/2} - 4f_{5/2}$	437.761	437.687 ± 0.020	437.694
"	$5g_{9/2} - 4f_{7/2}$	431.346	431.285 ± 0.017	431.283

3) the corrected values for the transition energies in keV. The uncertainties in the values of E_{th} in Table II range from 5 to 10 eV, and are the same as in Dixit *et al.* We see now by a comparison of the theoretical and experimental values that the discrepancy is reduced considerably, the only transitions having any significant discrepancy being the $4f$-$3d$ transition in Ba and the $5g$-$4f$ transition in Pb.

We have also considered the effect of the finite size of the nucleus on the vacuum-polarization corrections: The rather complicated convolution integral required to do this in configuration space becomes a trivial multiplicative factor in the momentum-space integral. Taking the uniform model for the nuclear shape, we find that the individual shifts are increased by rather less than 1%, giving less than 10 eV from the Uehling term itself, and correspondingly smaller corrections from the higher-order terms. A more realistic model for the nuclear shape can hardly be expected to give rise to any significant difference.

An effect of the same magnitude arises if the vacuum-polarization potential is included in the Dirac equation itself, rather than treating it perturbatively as we have done. The E_{th} in Table II is, in fact, calculated by the former method; the difference between this value and the perturbative value is of the order of 10 eV.

A number of authors[5,6] have suggested a possible anamolous interaction of the muon via a scalar field. The model suggested by Barshay,[5] with a scalar meson of mass around 750 MeV, turns out to give shifts of less than 1 eV for the states in which our interest lies; this is because the large mass implies a very short-range force which dominantly affects only the S-wave states. We note that the $4f$-$3d$ transition in Ba and the $5g$-$4f$ transition in Pb have almost equal discrepancies of about 70 eV. This enables us to put a rather good upper limit on the mass m_s of the scalar meson of about 8 MeV. Assuming this mass, the required coupling constant is $G_s = g_{s\mu\mu} \times g_{sNN} = 6 \times 10^{-7}$. The final column in Table II shows the theoretical value, including the effects of this particle, once again incorporating the effects of the finite nuclear size. It is remarkable that the remaining discrepancy is eliminated: The reader is at liberty to regard this as evidence for a physical particle of mass 8 MeV, coupling mainly to $\mu^+\mu^-$. It is amusing to speculate that if this particle is very weakly coupled to e^+e^-, it would escape experimentally detection, as well as provide a mechanism for the breaking of μe universality.

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EPR Investigations of Ce^{3+} in Cubic Sites of CaO, SrO, and BaO†

R. W. Reynolds,* Y. Chen, L. A. Boatner,* and M. M. Abraham
Solid State Division, Oak Ridge National Laboratory, Oak Ridge, Tennessee 37830
(Received 17 April 1972)

The EPR of Ce^{3+} has been observed in sites of cubic symmetry for the first time. The unusually large deviation of the observed g values for Ce^{3+} in CaO, SrO, and BaO from the theoretical value calculated for a pure Γ_7 ground state can be attributed mainly to static crystal-field admixtures.

The electron paramagnetic resonance (EPR) spectrum of Ce^{3+} ($4f^1$ configuration) in cubic single crystals has previously been the subject of numerous investigations.[1-6] Until the present work, however, no EPR spectra have been reported which could be attributed unequivocally to Ce^{3+} in a cubic symmetry site.[7] The observation of the EPR spectrum of Ce^{3+} in a site of local cu-