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Formation and Destruction of Magnetic Islands in Toroidal Systems

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We present a theoretical study of formation and destruction of magnetic surfaces in toroidal systems. As applied to levitrons the theory agrees with the numerical results.

Perturbed magnetic surfaces in toroidal systems are equivalent to nonlinear oscillating systems with rotational transform for frequency. Resonances that transform the unperturbed surfaces into a structure of magnetic islands we call *primary resonances*, and the *secondary resonances* transform the bound-state-like contours of a given island into a similar structure of secondary magnetic islands. To every magnetic island we attach two types of stochasticity, external due to the overlapping of primary resonances and internal due to the overlapping of secondary resonances (see Fig. 1).

1. Nonlinear oscillating systems.—We consider the nonlinear equations

$$dI/dt = (\epsilon/\nu)[\nu^{2\delta}\Gamma(I, \theta, t)] + O(\epsilon^2),$$
$$d\theta/dt = \nu\{1 + \epsilon[\nu^{2\delta'}\Pi(I, \theta, t)] + O(\epsilon^2)\}, \qquad (1.1)$$

where Γ and Π are periodic functions of θ and t, and ϵ is a small parameter. δ and δ' are introduced to allow Γ and Π to include terms that are not functions of ν. We derive Eqs. (1.1) for levitrons, where $2\pi\nu(I)$ is the rotational transform. (I, θ) are the action and angle variables corresponding to $(\frac{1}{2}r^2, \Phi)$, where r, Φ, and z are the toroidal coordinates, and t is determined from the relation $dt \equiv dz/B_z$. The major radius of the torus is normalized to 1. Equations (1.1) were also derived for the stellarator.[1]

1.1. Primary resonances: We let $\nu(I)$ be positive between the central magnetic axis and the separatrix. ($\nu < 0$ can be treated similarly.) Primary resonances are obtained from secular contributions and are characterized by $m_i\nu_i - l_i = 0$ where m_i, l_i are the lowest positive integers satisfying $m_i\nu_i - l_i = 0$. Their nonlinearity coefficients are given by

$$x_i = |d\nu(I_i)/dI| I_i/\nu_i. \qquad (1.2)$$

In the following, we assume that $x \gg \epsilon^{1/2}$. This assumption is verified for the levitron (refer to Table I).

In the vicinity of a given resonance stable is-

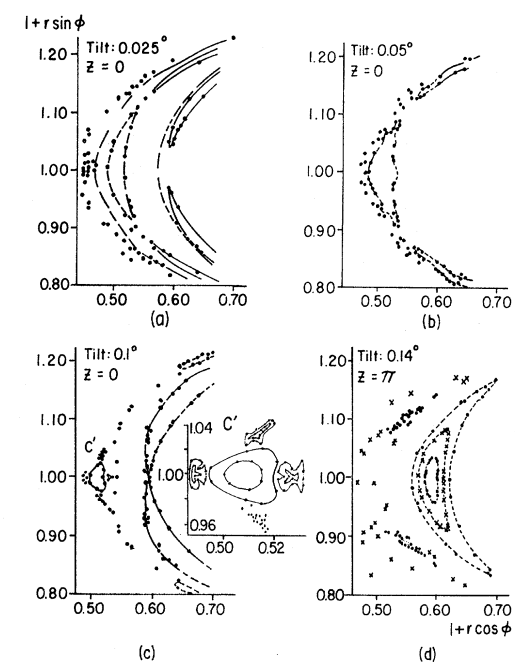

FIG. 1. (a) A closed contour of the π resonance. (b) The perturbation is doubled and the contour is heavily distorted. (c) The perturbation is doubled again, the contour is completely destroyed, and secondary magnetic islands appear. This is a typical example of destruction by internal overlapping. The 2π resonance after reaching its maximum flux in part (c) is seen partially destroyed in (d). (The numerical results are taken from Ref. 6.)

TABLE I. For the levitron ϵ is a tilt angle. The theoretical tilts for which the resonances are completely destroyed are approximately equal to the smallest of $\epsilon_j, \epsilon_{jj}$. The ϵ_c are the numerically measured critical tilts. The x_j represent the nonlinearity coefficients.

ν	x_j	ϵ_j (deg)	ϵ_{jj} (deg)	ϵ_c (deg)
3	1.256	(2.75)	5.70	2.50
5/2	1.074	(2.75)	4.35	2.00
2	1.045	4.95	(3.70)	3.00
3/2	1.140	1.68	(0.76)	0.70
1	1.300	0.54	(0.07)	0.30
1/2	2.200	0.54	(0.04)	0.15
1/4	2.280	$\sim$0.244	($\sim$0.01)	<0.02

land contours may occur in the region $\epsilon^{1/2}I_i \sim |\Delta I| \ll I_i$. When we expand (1.1) in this region and average over fast oscillations, we get

$$d\Delta I/dt = (2\epsilon/\nu_i) f_{\nu_i}(I_i, u), \qquad du/dt = \mu\, \Delta I, \tag{1.3}$$

where $u = l_i t - m_i \theta$, $\mu = -m_i d\nu(I_i)/dI$, and

$$f_{\nu_i}(I, u) = \frac{\nu_i^{2\delta}}{2\pi}\int_0^{2\pi} \Gamma(I, -u/m_i + \nu_i t, t)\, dt. \tag{1.4}$$

Equations (1.3) can be derived from the Hamiltonian

$$K(\Delta I, u) = \tfrac{1}{2}\mu(\Delta I)^2 - (2\epsilon/\nu_i)\int_0^u f_{\nu_i}(I_i, y)\, dy, \tag{1.5}$$

where surfaces of constant K will represent the island contours. The condition that ΔI be real defines a discrete set of intervals for u variations, thus forming m_i families of contours separated by a local separatrix. Each family we call an island, the center of which is an elliptic singularity. Relation (1.5) has been used to plot some magnetic island contours for the levitron.[2]

The primary resonance width Ω_i is related to the maximum action excursion by $\Omega_i = \nu_i x_i(\max|\Delta I|)/I_i$. We get

$$\Omega_i = 2\epsilon^{1/2}(x_i/I_i)^{1/2}\nu_i^{\delta} F_i(I_i, l_i, m_i), \tag{1.6}$$

where $F_i(I_i, l_i, m_i)$ is a form factor which characterizes the system and depends on the resonance parameters.

1.2. Island perturbation and secondary resonances: We let $\{\alpha_j\}_1^{m_i}$ be the set of elliptic singularities, and renormalize t to $\tau = \nu_i t$. The island oscillations [in terms of $v_j \equiv \Delta I(u(w_j))$, $w_j = (\operatorname{sgn}\mu)\tilde{u}_j$, where $\tilde{u}_j = u - \alpha_j$, and τ], are of the libration type. If we introduce the action and angle variables (J, η) corresponding to (v_j, w_j), the island contour equations become

$$dJ/d\tau = 0, \quad d\eta/d\tau = w, \tag{1.7}$$

where w is a positive function of J in the region between the elliptic singularity and the local separatrix.[3] At the local separatrix $w = 0$.

The island perturbations obtained from Eqs. (1.1) perturb (1.7) as follows:

$$\frac{d}{d\tau}\Delta J = \frac{(m_i x_i \epsilon^3)^{1/2}}{\omega(2I_i)^{1/2}\nu_i^{3-2e'}}\tilde{A}(J, \eta, \tau), \qquad \frac{d}{d\tau}\Delta\eta = \frac{d\omega}{dJ}\Delta J, \tag{1.8}$$

where $\tilde{A}(J, \eta, \tau)$ is a periodic function of η and τ,[3] and $e' = \delta + e$ and e is the lesser of δ and δ'.

Secondary resonances are characterized by $q_k\omega_k - p_k = 0$, where p_k, q_k are the lowest positive integers satisfying $q_k\omega_k - p_k = 0$. We let $M = -q_k \times d\omega(J_k)/dJ$, $\xi = p_k\tau - q_k\eta$, and $\tilde{f}_{\omega_k}(J, \xi) = \langle\tilde{A}(J, -\xi/q_k + \omega_k\tau, \tau)\rangle_\tau$. We linearize (1.8) near the secondary resonance ω_k (for $\epsilon I \sim |\Delta J| \ll \epsilon^{1/2}I$) and average over τ, and get

$$\frac{d\,\Delta J}{d\tau} = \frac{(2\epsilon^3 m_i x_i)^{1/2}}{\omega_k I_i^{1/2}\nu_i^{3-2e'}}\tilde{f}_\omega(J_k, \xi), \qquad d\xi/d\tau = M\Delta J. \tag{1.9}$$

Equations (1.9) can be derived from the Hamiltonian

$$h(\Delta J, \xi) = \tfrac{1}{2}M(\Delta J)^2 - \frac{(2\epsilon^3 m_i x_i)^{1/2}}{\omega_k I_i^{1/2}\nu_i^{3-2e'}}\int_0^\xi \tilde{f}_{\omega_k}(J_k, y)\, dy, \tag{1.10}$$

where surfaces of constant h represent the secondary island contours.

The secondary resonance width Δ_{ik} is related to the maximum excursion of J by $\Delta_{ik} = X_k\omega_k(\max|\Delta J|)/J_k$, where $X_k = |d\omega_k/dJ|J_k/\omega_k$. We get

$$\Delta_{ik} = \frac{\sqrt{2}}{\nu_i^{3/2-e'}}\left(\frac{2\epsilon^3 m_i x_i}{I_i}\right)^{1/4}\left(\frac{X_k}{J_k}\right)^{1/2}\tilde{F}_k(J_k, p_k, q_k), \tag{1.11}$$

where $\tilde{F}_k$ is a form factor that characterizes the system and depends on the secondary resonance parameters.

2. Instabilities and destruction of the magnetic surfaces.—2.1. Overlapping of resonances: It is well confirmed[4] that a strong instability with randomlike behavior occurs when resonances overlap. Overlapping of two neighboring resonances occurs if their separation is smaller or equal to the arithmetic average of their widths. Overlapping of resonances below a given frequency ν_j means that the sum of the widths of all resonances with frequencies smaller or equal to ν_j is greater or equal to ν_j (this assumes that overlapping proceeds in the order from the separatrix). We take this criterion to define the limit of stochasticity below ν_j; it is equivalent to

$$\nu_j \sim \sum_{\nu_i \leq \nu_j} \Omega_i(I_i, l_i, m_i), \tag{2.1}$$

and should give an underestimate of the critical perturbation for ν_j. Assuming that for every pair m_i, l_i there is a resonance, we get from (2.1)

$$\nu_j \sim \sum_{m=1/\nu_j}^{\infty} \sum_{l=1}^{m} [\Omega(I, l, m) - \sum_{p=2}^{\infty} \Omega(I, pl, pm)]. \tag{2.2}$$

If we replace sums by integrations (2.2) becomes

$$1 \sim \int_{1/\nu_j}^{\infty} m\, dm\, [\bar{\Omega}(m) - \int_{2m}^{\infty} dy\, \bar{\Omega}(y)], \tag{2.3}$$

where

$$\bar{\Omega}(m) = (1/\nu_j) \int_{1/m}^{\nu_j} \Omega(I, y, m)\, dy, \tag{2.4}$$

and $\Omega(I, l, m)$ is an interpolating function that equals Ω_i at the ith resonance position for all i's.

From (2.3) and (1.6), if ϵ_j is the limit of external stochasticity below ν_j, we get

$$G_j \epsilon_j^{1/2} \sim \tfrac{1}{2}(I_j/x_j)^{1/2} \nu_j^{-\delta}, \tag{2.5}$$

where G_j depends on the system and the ordering of the resonance.

If ϵ_{jk} is the limit of internal stochasticity for the ν_j resonance below the ω_k secondary resonance, from (1.11) and (2.3) we get

$$\tilde{G}_k \epsilon_{jk}^{3/4} \sim \frac{\sqrt{2}}{2} \left(\frac{I_j}{2m_j x_j}\right)^{1/4} \left(\frac{J_k}{X_k}\right)^{1/2} \nu_j^{3/2-e'}, \tag{2.6}$$

where $\tilde{G}_k$ depends on the system and the ordering of the secondary resonance ω_k.

Near the elliptic singularities we have shown that $X_k \sim 1$.[3] Also, from Eq. (1.5) one can easily show that $J_k \sim \epsilon_{jk}^{1/2} I_j \nu_j^{\delta-1/2} x_j^{-1/2}$. If we substitute in (2.6) we get

$$\tilde{G}_k \epsilon_{jk}^{1/2} \sim \frac{\sqrt{2}}{(2m_j)^{1/4}} I_j^{3/4} \nu_j^{5/4-(e+\delta/2)} x_j^{-3/4}. \tag{2.7}$$

Since each island is similar to the whole structure,[5] we assume that $\tilde{G}_j \sim G_j$. By substituting in (2.7) and (2.5) we get

$$\epsilon_{jj} = \frac{8}{(2m_j)^{1/2}} \left(\frac{I_j}{x_j}\right)^{1/2} \nu_j^{5/2-\delta''} \epsilon_j, \tag{2.8}$$

where $\delta'' = 2e - \delta$.

For the stellarator, where ϵ was below the external stochasticity limit,[1] formula (2.8) explains why destruction occurred near the separatrix. In Fig. 1, we show a typical example of destruction by internal overlapping in levitrons. We note also that secondary magnetic islands appear in Fig. 1(c).

For the levitron ($\delta = 0 = \delta'$) we calculated ϵ_j by directly testing the overlapping of neighboring primary resonances and deduced ϵ_{jj} by using Eq. (2.8). The results are tabulated in Table I. (For the levitron ϵ is a tilt angle; in Table I it is given in degrees.) We conclude that for $\nu_j \leq 2$ destruction is caused by internal overlapping. In Table I quantities in parenthesis are the theoretical limits for destruction. ϵ_c are the numerically measured tilts for which the resonances are completely destroyed.[6]

It has been established that if resonances overlap, a rapid destruction of their island structure occurs.[4]

(1) Thus, if primary resonances overlap, a rapid destruction of their flux surfaces is expected.

(2) For large nonlinearity coefficients ($x \gg \epsilon^{1/2}$), the field lines are trapped in an effective potential well near the primary resonances forming trapped contours in the region of each stable primary resonance. Another possible phenomenon of destruction is the overlapping of secondary resonances. Depending on the resonance (and the system) destruction may occur by either or both phenomena.

(3) The primary island width increases as $\epsilon^{1/2}$, while the secondary island width increases as $\epsilon^{3/4}$. Internal overlapping proceeds almost orderly from the local separatrix to the elliptic singularity. Therefore, for islands that are most affected by internal overlapping the *observed* primary width should increase at a slower rate than $\epsilon^{1/2}$ due to the successive disappearance of outer contours destroyed by secondary resonances overlapping. This is in agreement with the numerical observation by Freis *et al.*, where the 1 and $\frac{1}{2}$ resonance widths increase as $\epsilon^{1/2}$ until breakup while the $\frac{3}{2}$, 2, $\frac{5}{2}$, and 3 resonance widths increase as $\epsilon^{3/2}$.

(4) For small nonlinearity ($x \leq \epsilon^{1/2}$) the field lines may escape the resonance zone and cause instabilities.[3,7]

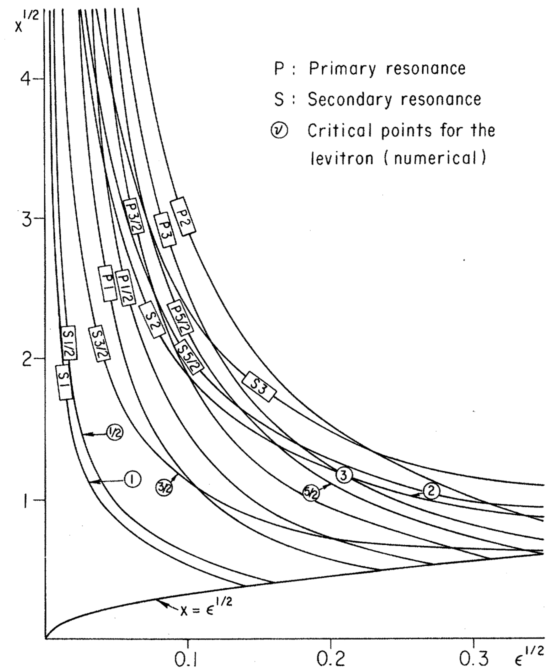

FIG. 2. Critical perturbations as functions of the nonlinearity coefficient for levitrons ($\delta = 0 = \delta'$). The regions of external (internal) stochasticity are on the right-hand side of the P (S) curves. Below the $x = \epsilon^{1/2}$ curve, the field line may escape the resonance zone. The encircled points are numerical (Ref. 6). The small arrows indicate the theoretical curves to which the numerical points belong. (ϵ is measured in radians.)

(5) For levitrons ($\delta = 0 = \delta'$) we plot the critical perturbations of resonances as functions of x (Fig. 2). These functions are determined from formula (2.5) [formula (2.7)] for primary (secondary) resonances. The arbitrary constants are determined from the values of x_j, ϵ_j (ϵ_{jj}) given in Table I. Obviously the regions of external (internal) stochasticity lie on the right-hand side of the P (S) curves. The encircled points are numerical, from Table I. We note that below $x = \epsilon^{1/2}$ there is no trapping in the sense described above.

(6) For systems which are characterized by $\delta'' < \frac{5}{2}$, Eq. (2.8) shows that the region of internal stochasticity extends over all of the x-ϵ plane as ν_j approaches zero; thus flux surfaces are always destroyed near the separatrix.

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Adiabatic Compression of the Tokamak Discharge*

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A tokamak discharge of $T_e \sim 1$ keV is produced in the absence of conducting-wall magnetohydrodynamic stabilization, and then compressed in major radius by a factor of 2.3 in a static toroidal field. Measurements are reported on the resultant rise of density, temperature, and plasma current.

The equilibrium major radius R of the tokamak discharge is determined by the Lorentz force exerted on the plasma currenty by the external vertical magnetic field B_z.[1,2] An increase of B_z can be used to effect compression of the discharge in R. For compression times short compared with the diffusion time, the flux of the (static) toroidal field B_t is conserved within the moving