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Nonlinear Evolution of the Rayleigh-Taylor Instability of a Thin Layer

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An exact, closed-form, nonlinear solution is obtained for the Rayleigh-Taylor instability of a thin fluid layer. If the layer evolves from a small sinusoidal perturbation, its subsequent shape is in the form of a cycloid. The solution is good up to a time t^* after which adjacent segments of the layer begin to collide with each other. A qualitative discussion of the layer dynamics for $t > t^*$ is given.

In this note we investigate the Rayleigh-Taylor instability of a thin fluid layer which is supported against gravity or inertial forces by the pressure of a massless fluid (e.g., a magnetic field). We shall idealize the layer by making the assumption that it is of infinitesimal thickness.[1] It turns out that the assumption of zero thickness will allow us to obtain a fairly complete description of the nonlinear evolution of the layer.

The present analysis was motivated by a proposed controlled-thermonuclear-fusion experiment (LINUS) to be conducted at the Naval Research Laboratory.[2] In this device a cylindrical conducting shell surrounds a θ-pinch plasma and is accelerated inward (imploded), thereby compressing the enclosed magnetic field and plasma. The great advantage of this scheme is that very large magnetic fields (several megagauss) and hence large plasma densities are attainable. This in turn makes possible a fusion device which can be much shorter than would be necessary for a fusion-power-producing θ pinch. Whether the Rayleigh-Taylor instability of the imploding shell can play a destructive role in this application depends on the amount of time that it is operative and on its nonlinear development.

We note that, while previous theoretical and numerical work[3-10] has concentrated on the nonlinear problem of the Rayleigh-Taylor instability of an interface separating two semi-infinite fluids, the problem of the nonlinear evolution of a layer appears to have received very little attention.

We consider a layer of fluid which, in equilibrium, is supported against gravity, $-g\hat{y}_0$, by a massless fluid of pressure p_a in $y<0$, with a second massless fluid of pressure $p_b = p_a - \sigma_0 g$ in $y>0$, where σ_0 is the surface mass density of the layer. For simplicity, the pressure difference $p_a - p_b$ is assumed to be independent of time as the layer evolves. Extensions of the present calculation to include time variation of the pressure difference and application of the method to a radially imploding cylindrical layer are given elsewhere.[11]

Let $y=0$, $x=\xi_0$ be the position of a point on the layer when it is in equilibrium. Assume that this equilibrium is perturbed at $t=0$ and then left to evolve. The new position of this point at time t is denoted by $\vec{r}(\xi_0, t) = x(\xi_0, t)\hat{x}_0 + y(\xi_0, t)\hat{y}_0$ (we restrict our analysis to two dimensions). Now consider another point whose equilibrium position is $y=0$, $x=\xi_0 + d\xi_0$. At time t this point will be located at $\vec{r} + (\partial\vec{r}/\partial\xi_0)\,d\xi_0$. We wish to find the equation of motion for this surface element which was originally located in the interval between ξ_0 and $\xi_0 + d\xi_0$. The mass of this element is clearly time independent and is given by $dm = \sigma_0 d\xi_0$. The force on the element is

$$d\vec{F} = -g\,dm\,\hat{y}_0 - (p_a - p_b)\,d\xi_0(\partial\vec{r}/\partial\xi_0)\times\hat{z}_0, \qquad (1)$$

where the first term in (1) is the gravitational force, and the second term is the pressure force, whose magnitude is just the pressure difference multiplied by the length of the element and whose direction is normal to the element (hence the cross product with $\hat{z}_0$). Thus since $dm\,\partial^2\vec{r}/\partial t^2 = d\vec{F}$ and $p_b - p_a = \sigma_0 g$, we obtain

$$\partial^2 x/\partial t^2 = -g\,\partial y/\partial\xi_0, \qquad (2)$$

$$\partial^2 y/\partial t^2 = g\,\partial x/\partial\xi_0 - g, \qquad (3)$$

with initial conditions x, y, $\partial x/\partial t$, and $\partial y/\partial t$ given at $t=0$. The remarkable thing about (2) and (3) is that they are linear. We immediately see that $x=\xi_0$, $y=0$ is a solution, and indeed this is just the unperturbed equilibrium. The most general solution of (2) and (3) is

$$x(\xi_0, t) = \xi_0 - \sum_{k,\sigma} \xi_x{}^\sigma(k,t)\cos(k\xi_0 + \theta_k{}^\sigma), \qquad (4)$$

$$y(\xi_0, t) = \sum_{k,\sigma} \xi_y{}^\sigma(k,t)\sin(k\xi_0 + \theta_k{}^\sigma), \qquad (5)$$

where the $\xi_{x,y}{}^\sigma(k,t)$ satisfy

$$\ddot{\xi}_{x,y}{}^\sigma + kg\xi_{y,x}{}^\sigma = 0 \qquad (6)$$

and σ labels the four linearly independent solutions of Eq. (6). Assuming $\xi_x{}^\sigma \sim \xi_y{}^\sigma \sim \exp(-i\omega t)$ we obtain $\omega^4 = k^2g^2$ which corresponds to two oscillatory roots, $\omega = \pm(kg)^{1/2}$, one unstable root, $\omega = i(kg)^{1/2}$, and one damped root, $\omega = -i(kg)^{1/2}$. These four roots agree with the linear analysis of Taylor[12] for a layer of finite thickness. We emphasize, however, that the solution (4) and (5) is nonlinear[13] in the Eulerian sense since a given k component does not correspond to a perturbation which is sinusoidal in x, y space. In order to consider the solution in more detail we look at a special case of (4) and (5):

$$x = \xi_0 - A_0\exp[t(kg)^{1/2}]\cos k\xi_0, \qquad (7)$$

$$y = A_0\exp[t(kg)^{1/2}]\sin k\xi_0. \qquad (8)$$

If we assume that $kA_0\exp[t(kg)^{1/2}] \ll 1$, then the second term in (7) can be neglected and (8) becomes $y \cong A_0\exp[t(kg)^{1/2})]\sin kx$, which is sinusoidal in space and corresponds to the linear solution obtained by Taylor.[12] For arbitrary amplitudes, Eqs. (7) and (8) are actually the parametric representation of a cycloid (i.e., the path followed by a point on the surface of a rolling wheel). As shown in Fig. 1, at $t=0$ the curve is approximately sinusoidal; at $t=t_1$ the sinusoidal shape has become distorted so that the maxima become broad and the minima become sharp; at $t=t^*$ a cusp develops in the curve; and at $t=t_2$ the curve becomes multivalued. For $t \geq t^*$ the solution (7) and (8) becomes unphysical since portions of the layer (e.g., points a and b in Fig. 1) must pass through each other in order to reach the state $t=t_2$ shown in Fig. 2. This indicates that at $t=t^*$ adjacent sections of the layer on either side of the cusp begin to collide with each other. The downward displacement of the cusp is $y=-1/k$ so that from (8) we obtain

$$t^* = (kg)^{-1/2}\ln(kA_0)^{-1}. \qquad (9)$$

The thin-layer approximation would not be expected to apply, even in the early phase of the evolution, unless the wavelength of the perturbation were large compared to the layer thickness. We note that although short wavelengths have the

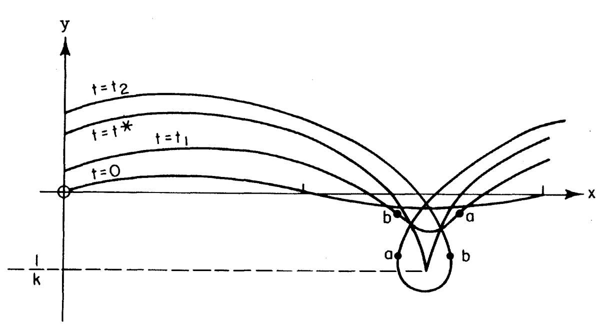

FIG. 1. Layer in the form of a cycloid.

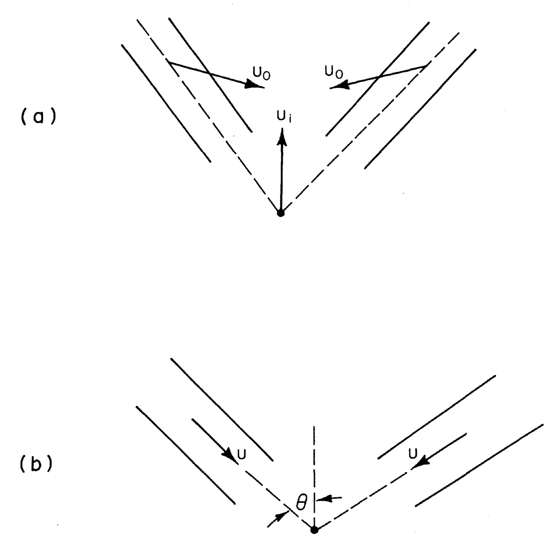

FIG. 2. (a) Colliding fluid layers; (b) the same in the frame of the intersection point.

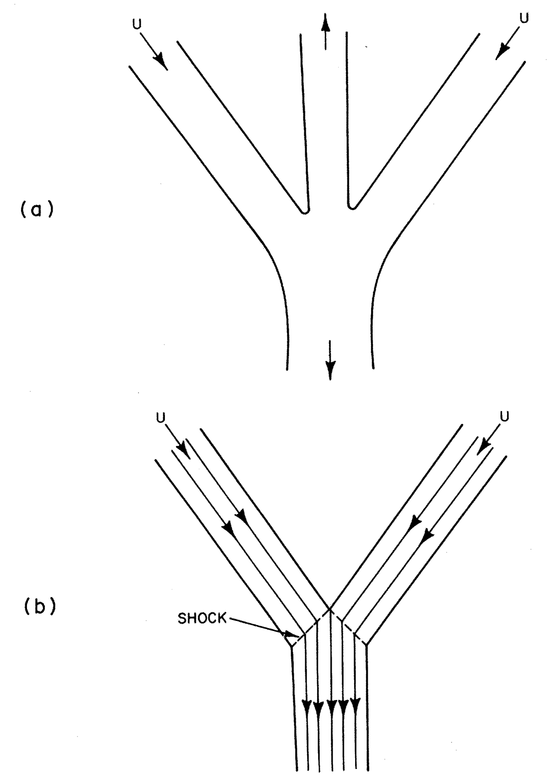

FIG. 3. Flow of the type studied in (a) Ref. 14 and (b) Ref. 15.

largest growth rates, it has been shown that long-wavelength components tend to dominate at later time.[3] This provides a partial justification of our approach even in the case in which the initial perturbation contains both long- and short-wavelength components.

We now consider the nonlinear evolution of the layer for $t > t^*$. Since Eqs. (7) and (8) imply that adjacent portions of the layer collide with each other for $t > t^*$, the question arises of what happens to two identical colliding slabs of fluid. Figure 2(a) illustrates the basic problem. In Fig. 2(a), U_0 is the velocity of the two fluid slabs and U_i is the velocity of the intersection point. Transforming to a frame moving with the intersection point we obtain the situation in Fig. 2(b). The problem in Fig. 2(b) has already been considered in some detail for the case of zero gravity where U and U_i are constant in time.[14,15] While these simplifying assumptions do not hold in our case, the solution of the simplified problem provides considerably insight. Basically two types of flow are possible.[14,15] The first type, pictured in Fig. 3(a), was found by Birkhoff, MacDougall, and Pugh[14] who obtained a complete solution of the problem for the case of an incompressible fluid. The second type of flow which can occur has been discussed by Walsh, Shreffler, and Willig[15] and is pictured in Fig. 3(b). In this solution, an attached shock forms at the point of joining of the two fluid streams, and only one emerging stream results. The solution of Walsh, Shreffler, and Willig is only possible when U is supersonic and the angle θ [defined in Fig. 2(b)] is sufficiently small, $\theta < \theta_c$. Otherwise the solution is of the type in Fig. 3(a). The critical angle θ_c depends in a complicated way on the equation of state, the fluid density, and U.[15] In the case where Fig. 3(a) applies, the evolution of the Rayleigh-Taylor instability for $t > t^*$ will be as shown in Fig. 4: The layer is in the form of a cycloid up to the point where adjacent sections of the layer collide; at this point the fluid from the previously colliding sections merge to form a downward-extending vertical layer which falls under the influence of gravity, and an upward jet of fluid. Under the influence of gravity, this upward jet splays out and falls back downward (like a fountain), splashing into the cycloidal segment. After the time when the splashing takes place it is difficult to predict the layer evolution. In the case where Fig. 3(b) applies there is no upward jet.

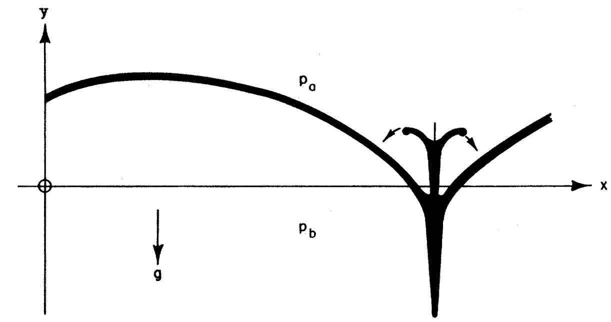

FIG. 4. Evolution of the layer for $t > t^*$ when the situation in Fig. 3(a) applies.

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[1]E. G. Harris, Phys. Fluids 5, 1057 (1962).

[2]J. P. Boris, R. Shanny, and N. K. Winsor, Bull. Amer. Phys. Soc. 17, 1029 (1972); J. P. Boris and R. Shanny, in Proceedings of the Fifth European Conference on Controlled Fusion and Plasma Physics, Grenoble, France, 1972 (to be published).

[3]E. A. Freiman, Astrophys. J. 120, 18 (1954).
[4]D. Layzer, Astrophys. J. 122, 1 (1955).
[5]G. Birkhoff and D. Carter, J. Math. Mech. 6, 769 (1957).
[6]P. R. Garabedian, Proc. Roy. Soc., Ser. A 241, 423 (1957).
[7]H. W. Emmons, C. T. Chang, and B. C. Watson, J. Fluid Mech. 7, 177 (1960).
[8]G. Birkhoff, in *Proceedings of the Thirteenth Applied Mathematics Symposium* (American Mathematical Society, Providence, R. I., 1960), p. 55.
[9]F. H. Harlow and J. E. Welch, Phys. Fluids 9, 842 (1966).
[10]B. J. Daly, Phys. Fluids 10, 297 (1967), and 12, 1340 (1969).
[11]E. Ott, to be published.
[12]G. I. Taylor, Proc. Roy. Soc., Ser. A 201, 192 (1950).
[13]Indeed a number of examples exist where problems which appear badly nonlinear in an Eulerian formulation have been simply solved by resorting to a Lagrangian formulation [J. M. Dawson, Phys. Rev. 113, 383 (1959); S. L. Ossakow, Ph. D. thesis, University of California at Los Angeles, 1966 (unpublished); R. C. Davidson and P. P. J. M. Schram, Nucl. Fusion 8, 183 (1968)].
[14]G. Birkhoff, D. P. MacDougall, and E. M. Pugh, J. Appl. Phys. 19, 563 (1948).
[15]J. M. Walsh, R. G. Shreffler, and F. J. Willig, J. Appl. Phys. 24, 349 (1953).

Anomalously Rapid Current Penetration in a Toroidal Turbulent Heating Experiment*

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The behavior of the plasma-current penetration in a toroidal device has been investigated for a wide range of plasma density ($10^{11} \lesssim n \lesssim 10^{13}$ cm^{-3}). The observed rapid penetration cannot be attributed to the enhancement of plasma resistivity. A possible explanation is proposed in terms of electrostatic instabilities driven by electron velocity gradients.

We report an experimental observation of anomalous skin effects in a toroidal system,[1] a schematic of which is shown in Fig. 1. An argon plasma with an average density $10^{10} \lesssim n \lesssim 10^{13}$ cm^{-3} is initially formed with the help of rf breakdown and preheating. The argon neutral pressure is typically 0.3 mTorr. A betatron-type electric field E ($\lesssim 100$ V/cm) is then applied to the plasma along a toroidal magnetic field B ($\lesssim 2$ kG). The maximum current induced in the plasma is about 20 kA at the density $n \simeq 10^{13}$ cm^{-3}. The electron temperature, measured by an orbit analyzer probe[2] and a diamagnetic probe, reaches several keV within 1 μsec starting from an initial temperature of the order of 10 eV. Poloidal magnetic fields produced by the plasma current were measured using small (5 mm diam) movable magnetic probes; one measures vertical magnetic fields and the other measures horizontal fields. It has been found that there is no significant difference in behavior between these two fields. Measurements were performed in the early stages of the current pulses, namely, up to 300 nsec, where skin effects, if any, should be most pronounced since the plasma resistivity is expected to be small.

The behavior of the electron acceleration by the electric field depends on the plasma density. At densities in the range $10^{10} - 10^{11}$ cm^{-3}, the electrons, as previously reported,[3] are freely accelerated up to about 100 nsec, when the electron drift velocity overtakes the electron thermal ve-

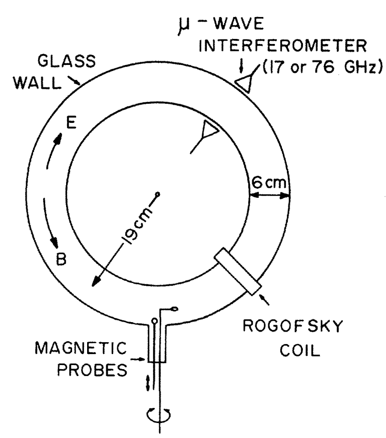

FIG. 1. Schematic of the device and major diagnostic tools. One magnetic probe measures vertical magnetic fields and the other measures horizontal fields.