Feynman-Kac Formula for Euclidean Fermi and Bose Fields

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We define free, covariant Euclidean Bose and Fermi fields and establish their relation with the corresponding relativistic free fields. Using this correspondence we prove a Feynman-Kac formula for boson-fermion models.

Euclidean boson fields play a fundamental role in the construction of Lorentz-covariant fields. Their importance was stressed by Schwinger[1] and Symanzik,[2] whose ideas led to an abstract formulation by Nelson.[3] Here we introduce free Euclidean fermion fields and the corresponding path-space formula for $\exp(-tH)$ and for Euclidean Green's functions.

Euclidean fermion fields involve complications absent for bosons.

(I) The Euclidean boson field $\Phi(x)$ agrees at time zero with the Lorentz boson field, $\Phi(0,\vec{x}) = \varphi(\vec{x})$. The Euclidean boson Fock space $\mathcal{E}_B$ then contains the Lorentz boson Fock space $\mathfrak{F}_B$.[3,4] This does not hold for fermions, i.e., $\mathfrak{F}_F \not\subset \mathcal{E}_F$, and sharp-time Euclidean Fermi fields create non-normalizable wave functions.

(II) Euclidean boson and fermion fields transform under the analytic continuation of the representation of the inhomogeneous Lorentz group to the inhomogeneous rotation group ISO_4 (Euclidean group). However, it is necessary to introduce two independent, anticommuting Euclidean fields Ψ^1 and Ψ^2 corresponding to the Lorentz fields ψ and $\overline{\psi}$. These extra degrees of freedom avoid a contradiction between Euclidean covariance of the fields ψ^1 and ψ^2, the canonical anticommutation relations, and the form of the two-point function, which has to be equal to the relativistic Feynman propagator at imaginary times.

(III) In contradistinction to the Euclidean boson action, the Euclidean action V involving fermions is non-Hermitian. The adjoint transformation $V \to V^*$ is related to Euclidean time inversion (see below). This non-Hermitian property causes no difficulty in the physical interpretation.

(IV) In spite of these differences, as for Euclidean boson fields, the action density for charge-conserving theories is Abelian. Thus our Feynman-Kac formula for fermion-boson systems gives a mathematically precise history integral for both fermions and bosons and relates the history (path-space) integral to a Hamiltonian.

In a separate publication,[5] a set of axioms for Euclidean Green's functions is given, and the axiomatic relation to Lorentz field theory is derived. Also a detailed version of the material presented here will be given elsewhere.[6] For notational conventions in the relativistic case, we mostly follow those of Bjorken and Drell.[7] We write x for a real four-vector $(x^0,x^1,x^2,x^3) = (x^0,\vec{x})$ and xy for the Euclidean inner product

$$\sum_{i=0}^{3} x^i y^i = x^0y^0 + \vec{x}\cdot\vec{y}.$$

On Euclidean Fock space $\mathcal{E} = \mathcal{E}_F \otimes \mathcal{E}_B$ we define two distinct Euclidean Fermi fields $\Psi_\alpha{}^1(x)$ and $\Psi_\alpha{}^2(x)$, $\alpha = 1,\ldots,4$, by

$$\Psi_\alpha{}^1(x) = (2\pi)^{-2}\sum_{j=1}^{4}\int e^{-ipx}(m_f{}^2+p^2)^{-1/2}[D(p,j)^*V_\alpha{}^j(p) + B(-p,j)U_\alpha{}^j(p)]\,d^4p,$$

$$\Psi_\alpha{}^2(x) = (2\pi)^{-2}\sum_{j=1}^{4}\int e^{-ipx}(m_f{}^2+p^2)^{-1/2}[D(-p,j)\hat{V}_\alpha{}^j(p) + B(p,j)^*\hat{U}_\alpha{}^j(p)]\,d^4p.$$

They satisfy the anticommutation relations

$$\{\Psi_\alpha{}^i(x),\Psi_\beta{}^j(y)\} = 0,$$

$$\{\Psi_\alpha{}^i(x),\Psi_\beta{}^j(y)^*\} = 2\delta_{ij}\delta_{\alpha\beta}(2\pi)^{-4}\int e^{-ip(x-y)}(p^2+m_f{}^2)^{-1/2}\,d^4p.$$

The two-point function is

$$\langle\Psi_\alpha{}^1(x)\Psi_\beta{}^2(y)\rangle = \langle \overline{T}\hat{\psi}_\alpha(x)\hat{\overline{\psi}}_\beta(y)\rangle = i\,S_{F\alpha\beta}(i(x^0-y^0),\vec{x}-\vec{y}),$$

where $\hat{\psi}$ and $\hat{\bar{\psi}}$ are the relativistic fields at imaginary time,

$$\hat{\psi}_\alpha(x^0,\vec{x}) = \exp(-x^0H_0)\psi_\alpha(\vec{x})\exp(x^0H_0), \quad \hat{\bar{\psi}}_\alpha(x^0,\vec{x}) = \exp(-x^0H_0)\bar{\psi}_\alpha(\vec{x})\exp(x^0H_0).$$

We also define the Euclidean Bose field $\Phi(x)$ by

$$\Phi(x) = (2\pi)^{-2}\int e^{-ikx}(k^2+m_b^2)^{-1/2}[A(k)^* + A(-k)]\,d^4k.$$

Then $[\Phi(x),\Phi(y)] = 0$ and

$$\langle\Phi(x)\Phi(y)\rangle = \langle T\hat{\varphi}(x)\hat{\varphi}(y)\rangle = i\bar{\Delta}_F(i(x_0-y_0),\vec{x}-\vec{y}),$$

where $\hat{\varphi}(x^0,\vec{x}) = \exp(-x^0H_0)\varphi(\vec{x})\exp(x^0H_0)$.

The Euclidean vacuum $\Omega_E \in \mathcal{E}$ is cyclic for the smeared fields $\Psi_\alpha{}^i(f),\Phi(g)$. There is a unitary representation $U_1(a)$ of the translation group and a unitary representation $U_2(A)$ of the universal covering group $SU(2)\otimes SU(2)$ of the four-dimensional rotation group SO_4, such that with $U(A,a) = U_2(A)U_1(a)$, for all $A \in SU(2)\otimes SU(2)$, $a \in \mathfrak{R}^4$,

$$U(A,a)\Omega_E = \Omega_E, \quad U(A,a)\Psi_\alpha{}^1(x)U(A,a)^{-1} = \sum_\beta R_{\alpha\beta}(A^{-1})\Psi_\beta{}^1(r(A)(x+a)),$$

$$U(A,a)\Psi_\alpha{}^2(x)U(A,a)^{-1} = \sum_\beta \Psi_\beta{}^2(r(A)(x+a))R_{\beta\alpha}(A), \quad U(A,a)\Phi(x)U(A,a)^{-1} = \Phi(r(A)(x+a)).$$

Here $A \to r(A)$ is the homomorphism of $SU(2)\otimes SU(2)$ onto SO_4 and $A \to R(A)$ is a four-dimensional unitary representation of $SU(2)\otimes SU(2)$.

As was explained in the introduction, the Lorentz boson Fock space $\mathfrak{F}_B$ is a subspace of the Euclidean space $\mathcal{E}_B$. This does not hold for fermions, and the relation between $\mathcal{E}_F$ and $\mathfrak{F}_F$ is more complicated.

In $\mathcal{E}$ we define the subspace $\mathcal{E}_+$, the subspace of "positive times," to be the closed linear hull of the set of vectors of the form

$$X = :\prod_{i=1}^{k}\Psi_{\alpha_i}{}^1(f_i)\prod_{j=1}^{k'}\Psi_{\beta_j}{}^2(g_j)\prod_{l=1}^{k''}\Phi(h_l):\Omega_E, \qquad (1)$$

where Wick ordering is defined as usual and the test functions f_i, g_i, and h_l are such that they vanish for $x^0 < 0$.

Now we define the mapping W_0 from a dense set of vectors in $\mathcal{E}_+$ onto a dense set of vectors in the Lorentz Fock space $\mathfrak{F}$ to be the linear extension of

$$W_0X = :\prod_{i=1}^{k}\hat{\psi}_{\alpha_i}(f_i)\prod_{j=1}^{k'}\hat{\bar{\psi}}_{\beta_j}(g_j)\prod_{l=1}^{k''}\hat{\varphi}(h_l):\Omega,$$

where X is defined by (1) and Ω is the vacuum vector in $\mathfrak{F}$. W_0 is a bounded operator. For a proof we define a unitary involution Θ on $\mathcal{E}$ such that $\Theta\Omega_E = \Omega_E$ and

$$\Theta\Psi_\alpha{}^i(x)\Theta^{-1} = \sum_\beta \Psi_\beta{}^{3-i}(\theta x)^*\gamma_{0\beta\alpha},$$

where $\theta x = (-x_0,\vec{x})$. Then for all vectors X,Y of the form (1), we find that

$$(W_0X, W_0Y)_{\mathfrak{F}} = (\Theta X, Y)_{\mathcal{E}}. \qquad (2)$$

Equation (2) shows that W_0 is a bounded operator with norm smaller than 1. Therefore, it can be extended to a bounded map W from $\mathcal{E}_+$ into $\mathfrak{F}$, and Eq. (2) holds for all $X, Y \in \mathcal{E}_+$ if we replace W_0 by W. We also find that for all $X \in \mathcal{E}_+$, $t \geq 0$,

$$WU_1((t,\vec{0}))X = \exp(-tH_0)WX, \qquad (3)$$

where $U_1((t,\vec{0}))$ is the unitary time translation group in $\mathcal{E}$. By formulas similar to (3) we can also establish the relation between the Euclidean fields and the Lorentz fields.

To establish the Feynman-Kac formula we consider a system with Yukawa interaction and polynomial boson self-interaction. Let P be a real polynomial of even degree with positive leading coefficient. In $\mathcal{E}$ we define the following operators:

$$\mathcal{P}_\kappa(t,V) = \int_0^t dx^0\int_V d^3x :P(\Phi_\kappa(x)):,$$

$$Q_\kappa(t,V) = \int_0^t dx^0\int_V d^3x \sum_\alpha :\Psi_{\kappa\alpha}{}^2(x)\Psi_{\kappa\alpha}{}^1(x):\Phi_\kappa(x).$$

The index κ denotes an ultraviolet cutoff; V is a finite volume in $\mathfrak{R}^3$.

Note that $Q_\kappa(t,V)$ is not symmetric, but using the involution Θ introduced above we have

$$Q_\kappa{}^*(t,V) = -\Theta Q_\kappa(-t,V)\Theta^{-1}.$$

Similarly, we define in the relativistic Fock space $\mathfrak{F}$

$$P_\kappa(V) = \int_V d^3x :P(\varphi_\kappa(\vec{x})):,$$

$$Q_\kappa(V) = \int_V d^3x \sum_\alpha :\bar{\psi}_{\kappa\alpha}(\vec{x})\psi_{\kappa\alpha}(\vec{x}):\varphi_\kappa(\vec{x}).$$

Now let $X \in \mathcal{E}_+$ be a finite linear combination of vectors of the form (1). Under the above assumptions, for $0 \leqslant t \leqslant \infty$,

$$W \exp\{-[\mathcal{P}_K(t, V) + Q_K(t,V)]\} U_1((t,\vec{0}))X = \exp\{-t[H_0 + P_K(V) + Q_K(V)]\} WX.$$

This is the Feynman-Kac formula. We remark that special care is needed for the definition of the operator $\exp[-Q_K(t, V)]$ because the exponent $Q_K(t,V)$ is not self-adjoint.

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Measurement of the Asymmetry Parameter in the Photoproduction of φ Mesons

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We have measured the asymmetry parameter $\Sigma = (\sigma_\parallel - \sigma_\perp)/(\sigma_\parallel + \sigma_\perp)$ for the photoproduction of φ mesons with photons polarized parallel and perpendicular to the plane of decay for the reaction $\gamma p \to \varphi p \to K^+K^-p$. We find $\Sigma = 0.985 \pm 0.12$ at a photon energy of 8.14 GeV and $|t|$ of 0.2 $(\mathrm{GeV}/c)^2$, consistent with pure diffraction production, or pure natural-parity Regge exchange.

As first pointed out by Freund and more recently by Barger and Cline,[1] φp elastic scattering is expected to proceed purely by Pomeranchukon exchange, and therefore photoproduction of φ mesons should be purely diffractive. If this is correct, polarized photons should produce φ's with polarizations 100% correlated with the incident polarization. Specifically, the asymmetry Σ, defined in terms of the yield of φ mesons $\sigma_\parallel$ produced with a polarization vector parallel to the incident photon polarization and the yield $\sigma_\perp$ normal to the incident photon polarization vector, should be unity[2]:

$$\Sigma = (\sigma_\parallel - \sigma_\perp)/(\sigma_\parallel + \sigma_\perp) = 1.$$

In the present experiment, the detection plane of the K pairs from the decay $\varphi \to K^+K^-$ is fixed perpendicular to the production plane, and the photon beam has a polarization which may be oriented perpendicular or parallel to the production plane. The measured asymmetry A is defined as $A = (N_\perp - N_\parallel)/(N_\perp + N_\parallel)$, where $N_\perp$ ($N_\parallel$) is the coincidence counting rate, corrected for accidentals, with the photon beam polarization vector normal (parallel) to the production plane. The quantity Σ is related to the measured asymmetry A by

$$\Sigma = A/|P_\gamma|(1-\epsilon),$$

where $|P_\gamma|$ is the magnitude of the photon beam polarization and ϵ is a small correction factor, about 6%, due to the finite angular acceptance of the K-pair spectrometer. In terms of density-matrix elements,

$$\Sigma = (\rho_{11}{}^1 + \rho_{1-1}{}^1)/(\rho_{11}{}^0 + \rho_{1-1}{}^1).$$