Intermediate-Range Gravity: A Generally Covariant Model

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A generally covariant classical model for gravity, which in the limit of weak fields is essentially equivalent to Fujii's massive dilaton theory, is obtained when the invariance under a group of space-time–dependent geometry-preserving mass-unit transformations of a scalar-tensor theory is broken by ascribing a mass to the scalar field. The experimental consequences are discussed.

In recent articles[1,2] Fujii has proposed a theory in which a nonzero mass is ascribed to the Nambu-Goldstone boson of scale invariance, the *dilaton.* The coupling of this particle with matter then leads to a gravitational potential V which has, in addition to the usual long-range Newtonian part, a component with a finite range of order m^{-1}, where $m\hbar$ (in units with $c=1$), is the dilaton mass. From arguments based on the dilaton-graviton mixing problem in strong-interaction physics, Fujii suggests[2] that m^{-1} should be of the order of 1 km or less. A study[1] of the existing experimental data limits m^{-1} to be either between 10 m and 1 km, or less than 1 cm.

It is the purpose of this note to present a purely classical (nonquantum) generally covariant model for gravity which, in the weak-field approximation, reduces to a theory essentially equivalent to Fujii's. It is hoped in this way not only to extend the validity domain of the theory beyond weak fields but also to gain some insight into the meaning of scale invariance and units transformations and their connection with gravity.

In a previous work[3] the relationship between Mach's principle and a group of space-time–dependent scale changes in the unit of mass was discussed within the context of the Brans-Dicke theory[4] of gravity. Transformations of the scalar field φ and the stress-energy tensor T_{ij} under this (mass-gauge) group were defined so as to preserve both (i) the geometry of space-time and (ii) the form of the field equations and hence the law of conservation of energy. A theory fully covariant under this group is obtained when the Brans-Dicke coupling constant $\omega=0$. The strange situation then results where a knowledge of the space-time geometry (e.g., as deduced from astronomical observations) does not suffice to determine, even qualitatively, the mass-energy content.[3]

We now consider the effect of breaking mass-gauge invariance by adding a scalar-field mass term $-m^2f(\varphi)$ to the Lagrange function. This results in the action principle

$$\delta\int(-g)^{1/2}[\varphi R+16\pi L-m^2f(\varphi)]\,d^4x=0\,, \tag{1}$$

with R the curvature scalar, L the Lagrangian for matter. The field equations are

$$G_{ij}=8\pi\varphi^{-1}T_{ij}+\varphi^{-1}\times(\varphi_{;ij}-g_{ij}\Box\varphi-\tfrac{1}{2}m^2fg_{ij})\,, \tag{2}$$

$$3\Box\varphi-m^2(\varphi f'-2f)=8\pi T\,, \tag{3}$$

where a semicolon denotes covariant differentiation, a prime denotes differentiation by φ, $\Box\varphi=g^{ij}\varphi_{;ij}$ with g_{ij} the metric (signature +2), and G_{ij} is the Einstein tensor. The *scalar* property of φ prevents us from specifying *a priori* the form of the function f. We note also that the weak principle of equivalence[4] (i.e., geodesic motion for small particles) remains valid for the action (1).

For the treatment of local and astronomical (noncosmological) problems we demand that a weak-field approximation should be possible. For this purpose we write

$$g_{ij}=\eta_{ij}+h_{ij},\quad \varphi=\varphi_0+\xi\,, \tag{4}$$

where η_{ij} is the flat-space metric, $\varphi_0=G_0^{-1}$ is the constant background value for the scalar field, and h_{ij} and ξ are small perturbations due to local masses. Consistency of the field equations then requires that $f(\varphi_0)=f'(\varphi_0)=0$. Because of the factor m^2, we can, without loss of generality, set $f''(\varphi_0)=3\varphi_0^{-1}$. The simplest such function is given by

$$f(\varphi)=3(2\varphi_0)^{-1}(\varphi-\varphi_0)^2\,. \tag{5}$$

However, the weak-field equations, which are our particular interest in this note, are independent of the precise form of f.

Assume now a stationary mass distribution. Keeping only terms of first order in h_{ij} and ξ, using the coordinate conditions

$$\eta^{jk}(h_{ij} - \tfrac{1}{2}\eta_{ij}h)_{,k} = G_0 \xi_{,i}, \qquad (6)$$

where a comma indicates partial differentiation, and solving the resulting (weak field) equations, we find

$$\varphi = \varphi_0 - \tfrac{2}{3}\int (Te^{-mr}/r)\, d^3x' , \qquad (7)$$

$$g_{ij} = \eta_{ij} + 4G_0 \int (T_{ij}/r)\, d^3x' - 2G_0\eta_{ij}\int (1 - \tfrac{1}{3}e^{-mr})(T/r)\, d^3x' , \qquad (8)$$

where $r = |\vec{x} - \vec{x}'|$ and the integrations are over all three-space. For time-dependent systems the above can be replaced by the corresponding retarded-time solutions. Note that putting $m \to \infty$ recovers precisely the general relativistic expressions.[5]

We consider next a point source of mass M. From (8) the g_{00} component of the metric yields the modified gravity potential

$$V(r) = -(G_0 M/r)(1 + \tfrac{1}{3}e^{-mr}). \qquad (9)$$

This is exactly the form used by Fujii[1] to illustrate the experimental consequences of his theory. Thus if m^{-1} does lie between 10 m and 1 km, G_0 will be the gravity constant involved in geological and planetary phenomena, while that measured by a Cavendish experiment will be $G = \frac{4}{3}G_0$. Therefore the same renormalization by the factor $\frac{4}{3}$ of the planetary masses will apply in our model.

The solution (8) furthermore yields expressions for the gravitational red shift

$$\delta\nu/\nu = \delta[(G_0 M/r)(1 + \tfrac{1}{3}e^{-mr})], \qquad (10)$$

and for the deflection of a light ray passing (e.g., the sun) at a minimum distance R from the center,

$$\theta = 4G_0 M R^{-1}. \qquad (11)$$

Only for values of $r \sim m^{-1}$ does (10) differ from the usual Einstein value. For the light deflection, on the other hand, the intermediate range force has, to this approximation, no effect at all. Finally, to discuss the perihelion shift, we need g_{00} to second order in r^{-1}. Nevertheless it is clear that if $r > m^{-1}$, as is the case for Mercury, the general relativistic value will result.

The almost-flat aspect of space in our immediate vicinity is usually attributed, *via* Mach's principle, to the effect of all the distant matter in the universe. Similar arguments can be advanced to account for the background value G_0^{-1} of the scalar field since it is well known that, approximately, $G_0 = \mathcal{R} M^{-1}$, where $\mathcal{R}$ is the radius and M the mass of the visible universe. One of the more satisfactory attributes of the present model is that, besides the range m^{-1} of the non-Newtonian force (which value, Fugii claims,[2] should be derivable from elementary-particle physics) no additional parameters have been introduced.

Finally, a word on scale invariance: The most natural extension of this symmetry to curved space is *via* conformal invariance. However, it can be shown[6] that adding a symmetry-breaking term to the conformal invariant scalar-tensor theory cannot lead to a theory incorporating an intermediate-range force. The close agreement between the predictions of the model advanced in this note and Fujii's massive-dilaton theory suggest that a close connection might exist between scale invariance and the group of geometry-preserving space-time–dependent transformations[3] of the unit of mass.

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