Nonlinear Mode Coupling and Relaxation Oscillations*

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By means of simple model equations the mode coupling between linearly unstable and stable waves is investigated. A sharp relaxation oscillation, damped either in the presence of spontaneous emission, or by a quasilinear reduction of the growth rate, is found. Effects of external modulation of the growth rate are investigated. The final saturation level exhibits two-threshold behavior as a function of the growth rate.

One of the important problems in the theory of plasma wave instabilities is to investigate how the growth of unstable waves can be saturated by nonlinear wave interactions. A number of attempts have been made to derive the final saturation state. Typical of these is the use of weak-turbulence theory which assumes a stationary balance between linear growth and nonlinear damping due to mode coupling of waves. Little is known, however, about the temporal development of a stationary state starting from the initial linear growth.

In this paper, we consider a simple model which we believe describes some basic features of the mode coupling between growing and damped waves, and investigate the transient behavior in approaching a stationary saturation state with the aid of a numerical computation. We find several interesting features, such as "two-threshold" behavior of the final saturation level as a function of the growth rate, a sharp relaxation oscillation in the absence of the source term, an efficient damping of the relaxation oscillation by a source term as well as by a quasilinear reduction of the growth rate, and excitation of the relaxation oscillation by an external modulation of the growth rate.

Construction of the model.—Let us consider a resonant three-wave decay process. The kinetic equation describing this process has been derived by many authors within the framework of weak-turbulence theory.[1] We consider the case where waves are linearly unstable in a relatively narrow region in $\vec{k}$ space. As the unstable waves grow, they decay into linearly stable waves. The wave-number region into which the unstable waves can decay is determined by the frequency and wave-number matching conditions, as well as the strength of the decay interaction coefficient. If these conditions are not very stringent, the region into which the unstable waves decay can be much larger than the unstable region. Let us divide the $\vec{k}$ space into subregions, with the linearly unstable region being one of them, such that both the linear damping rate and the interaction coefficient change very little over each subregion. Denoting the occupation-number density of the waves in the stable subregion i by H_i and that in the unstable region by I, we can write the rate equations for I and H_i in the following form:

$$\frac{\partial I}{\partial t}=2\gamma I+S+\sum_{(ij)}W_{ij}[H_iH_j-I(H_i+H_j)],\tag{1}$$

$$\frac{\partial H_i}{\partial t}=-2\nu_iH_i+R_i+W_{ij}[I(H_i+H_j)-H_iH_j]+\sum_{[m,l]}W_{ml}{}^i[H_mH_l-H_i(S_mH_l+S_lH_m)],\tag{2}$$

where γ and ν_i are the linear growth and damping rates of I and H_i; S and R_i are their spontaneous emission rates; (i,j) denotes the pair of subregions for which the decay conditions of I (decaying into H_i and H_j) are satisfied; W_{ij} is the interaction coefficient for this decay process; and the last term on the right-hand side of (2) represents the three-wave process among the stable waves (H_i, H_m, and H_l), that is, the secondary decay and its inverse process, S_l being $+1$ if the frequency ω_l of H_l is less than ω_i and -1 if $\omega_l>\omega_i$. We note that the mode represented by H_i may be different from the mode which is unstable.

We now simplify these equations, first by replacing γ_i, R_i, W_{ij}, and $W_{l,m}{}^i$ by constant values and then by approximating $\sum_{(i,j)}H_iH_j$ by bH^2 with a constant coefficient b, where H $(=\sum_iH_i)$ is the total occupation-number density of stable waves in the decay region. Equations (1) and (2) can then be reduced to the following form:

$$\partial I/\partial t=2\gamma I+S+W(bH^2-IH),\tag{3}$$

$$\partial H/\partial t=-2\nu H+R+2W(IH-bH^2)+\alpha H^2,\tag{4}$$

where the term αH^2 represents the effect of mutual interactions among the linearly stable waves

Making the transformation

$$H \to (\nu/W)H, \quad I \to (\nu/2W)I,$$
$$\partial/\partial t \to 2\nu\, \partial/\partial t, \quad R \to (2\nu^2/W)R, \tag{5}$$
$$S \to (\nu^2/W)S, \quad \nu \to \nu g,$$

one finally obtains the following model equations:

$$\partial I/\partial t = gI - IH + S + bH^2, \tag{6}$$

$$\partial H/\partial t = -H + IH + R + (\alpha - b)H^2. \tag{7}$$

Let us estimate the various coefficients which appear in (6) and (7). First we note that if the number N of subregions in the decay region is large, i.e., $N \gg 1$, then $b \sim 1/N \ll 1$ and $R \sim NS \gg S$. We also expect α to be very small since a balance will be maintained between the secondary decay and its inverse process. In order to estimate R (or S), let us consider the case of decay of an unstable ion wave into a couple of lower-frequency stable ion waves. Though this process is prohibited if one strictly uses the frequency and wave-number matching conditions, it is allowed in the long-wavelength region by correlation broadening.[2] Neglecting the higher-order terms in $k^2\lambda_D^2$ (λ_D is the electron Debye length), we have $W_{12} = \omega_1\omega_2\omega(\Delta k)^3/(8\pi n T_e \Delta\omega)$, where $\omega = \omega_1 + \omega_2$, $\Delta\omega$ is the characteristic resonance width, and $(\Delta k)^3$ is the volume in $\vec{k}$ space of a subregion. Noting that the spontaneous emission rate for a single wave is $2\nu T_e/\omega$ and that $N(\Delta k)^3 \sim \lambda_D^{-3}$, we find

$$R \sim N \frac{W}{\nu^2}\frac{2\nu T_e}{\omega} \sim \frac{1}{4\pi n \lambda_D^3}\frac{\omega_1\omega_2}{\nu\Delta\omega} \tag{8}$$

which is usually very small.

Equations (6) and (7) can be used equally well as model equations to describe mode coupling by nonlinear Landau damping. In this case both b and α vanish, since there is no nonlinear source or change of total occupation number. One can also allow a time dependence of the growth rate in Eq. (6). The time dependence may be due to either a quasilinear effect or an external modulation. Detailed treatment of these effects will be described below.

Solution of the model equations (6) and (7). —We first consider the case of constant growth rate. In the absence of the sources ($S = R = b = \alpha = 0$), we have two steady-state solutions of (6) and (7): $I(t) = 1$, $H(t) = g$, and $I(t) = H(t) = 0$. Of these only the former is stable for $g > 0$. Small-amplitude motion about this stationary state is sinusoidal with frequency $\omega_0 = \sqrt{g}$. Taking the

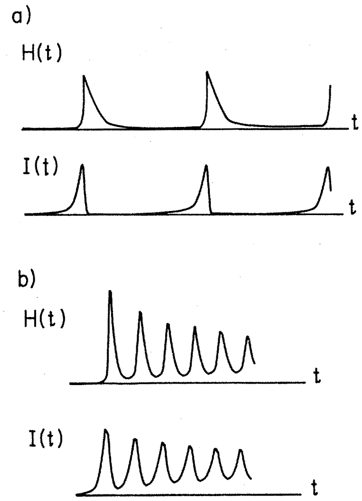

FIG. 1. (a) $H(t)$ and $I(t)$ in the case of large growth rate $g = 1$ and no damping; (b) with smaller growth rate $g = 0.5$ in the presence of linear sources $R = 10^{-2}$, $S = 10^{-3}$. In (a) $H_{max} \cong I_{max} \simeq 10$ and the period $\tau \approx 15$, to be compared with the small-amplitude period of $\tau = 6.28$. In (b) $H_{max} \approx 2$, $I_{max} \approx 3$, and $\tau \approx 10$, to be compared with the small-amplitude period of $\tau = 9$.

ratio of Eqs. (6) and (7) with $S = R = b = \alpha = 0$, we find $dH/dI = (I-1)H/I(g-H)$, from which an H-I phase diagram of the motion can be readily constructed.[3] The motion is periodic and characteristically the growth and decay are quite steep if we start from small initial values of I and H. An example of H and I as functions of time is shown in Fig. 1(a).

The introduction of the linear source terms, R and S, changes the steady-state levels to

$$H_0 = \tfrac{1}{2}\{g + R + S + [(g - R + S)^2 + 4RS]^{1/2}\}, \tag{9}$$

$$I_0 = \{g - R - S + [(g - R + S)^2 + 4RS]^{1/2}\}(2g)^{-1}. \tag{10}$$

The solution is now unique and Fig. 2 gives its behavior for the case $R = 10^{-2}$ and $S = 10^{-3}$. Considered as a function of g, I_0 shows weak threshold behavior near $g = 0$, the slope being $dI_0/dg = RS/(R+S)^3$, whereas H_0 is relatively unaffected. Near $g = R$, I_0 displays strong threshold behavior, increasing practically to its limiting value of 1 by the time g is a few times greater than R. When $g > R$, H_0 grows linearly with a constant slope of $dH_0/dg \simeq 1$.

Another interesting effect of the source terms is to produce damping of the relaxation oscillations. The damping β and modified frequency ω_1 are given by $2\beta = H_0 - g + 1 - I_0$ and $\omega_1^2 = \omega_0^2 - \beta^2$,

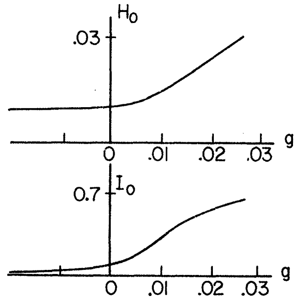

FIG. 2. The steady-state values H_0 and I_0 as functions of the growth rate g. In this example the linear source $R=10^{-2}$ and thus the threshold for I_0 is $g=10^{-2}$.

where $\omega_0^2=I_0g-g+H_0$. In the limit of large growth rate, i.e., $g\gg R$, these reduce to $2\beta\simeq S+R/g$, and $\omega_0^2\simeq g+S-R$. The damping causes the trajectory in the H-I plane to spiral in to the values H_0, I_0. Examples are shown in Fig. 3 for two different growth-rate values. Because of the threshold behavior of I_0 near $g=R$, a small change in g can cause large-amplitude changes in I_0. The actual behavior of I and H as functions of time starting from initial small values consists of a large overshoot followed by damped oscillations about the steady-state level. An example is shown in Fig. 1(b).

The effect of the nonlinear source terms, bH^2 and $(\alpha-b)H^2$ in Eqs. (6) and (7), is primarily that of changing the damping of the relaxation oscillations. The steady-state values are also changed to become $H_0\simeq g(1+bg)+S$, $I_0\simeq 1-R/g+(b-\alpha)g$, and the damping becomes $2\beta=S+R/g+gb(1+g)-\alpha g$. If αg is so large as to overcome the damping, the steady state becomes unstable and our model is no longer appropriate.

Let us now investigate the effect of a quasilinear reduction of the growth rate. We consider the case in which the growth rate for mode I is provided for by the amplitude P of another (pump) wave (e.g., an electromagnetic wave in the case of parametric excitation), i.e., $g=cP-d$, where we include the linear damping d of I. The rate equation for P may be written in the form

$$dP/dt=-\Gamma P+w-cPI, \tag{11}$$

where Γ is the linear damping constant for P and w is a constant source. For steady state $dP/dt=0$, we find

$$g=a(1+I/v)-d, \tag{12}$$

where $a=cw/\Gamma$ and $v=\Gamma/c$. The small-amplitude oscillations about the steady-state level now have

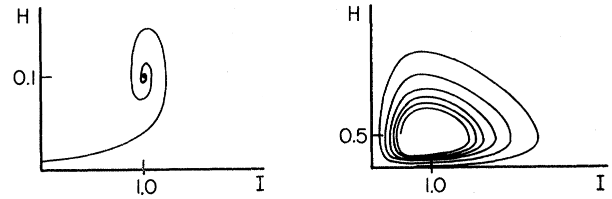

FIG. 3. The trajectory in the H-I plane with linear sources, $R=10^{-2}$, $S=10^{-3}$, for small and large growth rate, g. (a) $g=0.1$, (b) $g=0.5$.

a damping given by

$$2\beta\simeq\frac{(d-a)S}{R}+\frac{R+S}{a-d}+\frac{2a}{v},$$

where we have used $S/R\ll 1$, $R\ll a-d$. Thus large damping is produced by quasilinear reduction of the growth rate (i.e., pump depletion) when the term $2a/v$ dominates over the damping due to the sources.

Let us finally investigate the effects of external modulation of the growth rate, i.e.,

$$g=g_0[1+z\cos\Omega t]. \tag{13}$$

If the modulation frequency Ω is tuned to twice the natural frequency ω_0 of the relaxation oscillation, parametric amplification is expected to occur at a threshold value given by $g_0z=2R/g_0+O(R^2/g_0^2)$. Figure 4 shows the response of the system to modulation at the frequency $\Omega=2[g_0-R-(R^2/4g_0^2)]^{1/2}$ with b and α being set equal to 0. Plotted are the Fourier components of the amplitude of H at frequencies of $\Omega/2$, the natural frequency of the system, and for comparison also at Ω, the frequency of the modulation. The system is initially near the steady-state values

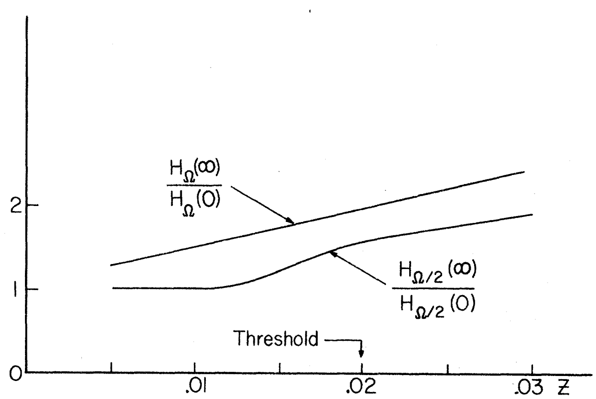

FIG. 4. Response to pump modulation at frequency Ω. The forced oscillations are linear in z; the response at frequency $\Omega/2$ shows a threshold. In this example $\Omega=2$, $g=1$, $S=10^{-3}$, $R=10^{-2}$.

and at $t=0$ the modulation begun. Note that the response of the modulation frequency is linear in the modulation amplitude z, whereas the response at the natural frequency shows threshold behavior near the theoretical value of z. The amplification saturates at a low level possibly because of detuning produced by nonlinear frequency shift.

The linearly driven oscillations produced by the growth-rate modulation exhibit a resonance behavior near $\Omega=\omega_0$, the natural frequency. Linearizing with respect to z and neglecting the nonlinear sources, we find the amplitude of the oscillation of H at the modulation frequency to be

$$|\Delta H_\Omega| = \frac{g_0^2 z}{2[(\Omega^2 - g_0 + R)^2 + \Omega^2(S+R/g_0)^2]^{1/2}}$$

and thus $|\Delta H_\Omega|_{\max} \cong g_0 z/[2(S+R/g_0)]$ which can be very large.

In conclusion, by analyzing the model equations (6) and (7) we find many interesting features regarding the saturation states, relaxation oscillations and their damping. Although our model equations represent a somewhat idealized situation they will be a good first approximation for a number of different problems in which mode coupling between growing and damped waves plays an important role. However, our equations do not include the effects of harmonic generation. Since harmonically generated waves are coherent, it is inappropriate to include them in H. Inclusion of this effect will be left to a future investigation.

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[1]See, for instance, V. N. Tsytovich, *Nonlinear Effects in Plasmas* (Plenum, New York, 1970), Chap. 5.

[2]V. N. Tsytovich, Plasma Phys. 13, 741 (1971).

[3]See, for instance, H. T. Davis, *Introduction to Nonlinear Differential and Integral Equations* (Dover, New York, 1962), pp. 102 ff.

Coherent Quasiparticle Excitation in a Type-II Superconductor in Crossed Electric and Magnetic Fields

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Structures are observed in the current-voltage characteristics of a type-II superconductor in crossed fields. This effect can be explained as due to coherent quasiparticle excitation when the electromagnetic fields satisfy excitation conditions.

In previous reports the author showed that in a longitudinal magnetic field some structures are observed in the magnetic field dependence of the superconducting current threshold[1,2] and in the current-voltage characteristics of the resistive superconducting state[2] of a type-II superconductor. In this Letter we would like to show that in a transverse magnetic field, structures are also observed in I-V curves, and that the origin of the structures can be attributed to coherent quasiparticle excitations which occur when certain conditions for the fields are satisfied.

We investigated the properties of the resistive superconducting state of Nb-25% Zr wire specimens of 0.025 cm diam at 10°K in a transverse magnetic field H and pulsed electric field E. Figure 1 shows a typical example of the current-voltage curve. Steplike structures appear in the curves similar to those seen in the longitudinal magnetic field.[2] In Fig. 2 the values of the fields E and H where the structures appear are plotted. It is seen from the figure that the values of the