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Critical Exponents for Charged and Neutral Bose Gases above λ Points*

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Critical behavior and exponents above the λ-transition points in charged and neutral Bose systems are examined with emphasis on the correlation of density fluctuations. Exponents are calculated with Wilson's expansion methods for arbitrary d to $O(1/m)$, where d is the dimension and m the number of components of the Bose field. Some results for arbitrary m to $O(\epsilon^2)$ ($\epsilon = 4-d$) are also obtained.

I. *Introduction.*—The behavior near the λ-transition temperature T_c of a charged Bose gas has been of considerable theoretical interest.[1,2] In this note we examine the critical behavior above T_c with the newly developed methods of Wilson.[3,4] We report results on critical exponents for arbitrary d to $O(1/m)$, and also those for arbitrary m to $O(\epsilon^2)$ ($\epsilon \equiv 4-d$). Here d is dimension and m is the number of components of the (complex) Bose field. Those readers who are interested only in exponents for neutral systems may go directly to Sect. II and then skip Sect. III.

A charged Bose gas is a system of bosons of the same charge. A uniform rigid background of the opposite charge is imposed to assume total charge neutrality. We shall avoid time-dependent effects and start with a model of a static m-component Bose field $a_{j\vec k}$, $j=1, 2, \ldots, m$, interacting with a static Coulomb field $c_{\vec k}$. By choosing appropriate units, the Hamiltonian is

$$\mathcal{H} \equiv \frac{H-\mu N}{T} = \sum_{\vec k}\left[(r_0+k^2)\sum_{j=1}^{m} a_{j\vec k}^* a_{j\vec k} + k^2 c_{\vec k}^* c_{\vec k} + \frac{e}{\sqrt{2}}\rho_{\vec k}(c_{\vec k}^* + c_{-\vec k})\right]. \tag{1}$$

Within a small temperature range near T_c, r_0 is just $-\mu/T$, and all other parameters may be taken as constants. $\rho_{\vec k}$ is the Fourier transform of the boson density

$$\rho_{\vec k} \equiv \sum_{\vec p}\sum_{j=1}^{m} a_{j\vec p}^* a_{j\vec p+\vec k}, \tag{2}$$

and the effect of the charged background is accounted for by imposing the restriction

$$c_{\vec k} = 0, \text{ if } k=0. \tag{3}$$

The terms in (1) involving $c_{\vec k}$ are designed to reproduce the usual $e^2\rho_{\vec k}\rho_{\vec k}^*/k^2$ Coulomb interaction between bosons.[5] The reason for introducing $c_{\vec{k}}$ as an additional field will be clear later. The quantities of interest are the density correlation function χ and the Green's function G:

$$\chi(k) = \langle \rho_{\vec{k}} \rho_{\vec{k}}^* \rangle, \quad G(k) = \langle a_{\vec{k}} a_{\vec{k}}^* \rangle. \tag{4}$$

The statistical average $\langle \cdots \rangle$ is taken over the density matrix [not to be confused with $\rho_{\vec{k}}$ in (2)]

$$\rho \propto \exp(-\mathcal{H}). \tag{5}$$

The Coulomb force becomes infinite as $k \to 0$. Let us define χ', the irreducible part of $\chi(k)$, by excluding isolated Coulomb lines of momentum k:

$$\chi = \chi'(1 + e^2\chi'/k^2)^{-1}. \tag{6}$$

For formal discussion, we define the Coulomb-field propagator $D(k)$ as

$$D = (k^2 + e^2\chi')^{-1}. \tag{7}$$

$e^2\chi'$ is thus the "self-energy" for the Coulomb field. As a result of the restriction (3), the isothermal compressibility[6] is given by $\chi'(0)$, not by $\chi(0)$, and the specific heat is $C_p \sim \chi'(0) + \text{const}$ near T_c.

II. *Results.*—The exponents η and γ for a charged Bose system are shown to be the same as those for a neutral Bose system. The difference between the charged and neutral cases appears through the density correlation functions. Let $\chi_n(k)$ be the density correlation function for a neutral Bose system, then we find for small and for zero k

$$\begin{aligned} &\chi_n(k) \sim \text{const} - \operatorname{sgn}(\lambda) k^\lambda, \text{ at } T_c, \\ &\chi_n(0) \sim \text{const} + \operatorname{sgn}(\alpha)(T - T_c)^{-\alpha}, \quad T > T_c. \end{aligned} \tag{8}$$

For the charged case, we have

$$\chi(k) \sim k^2 \tag{9}$$

always, and

$$\begin{aligned} &\chi'(k) \sim \text{const} - k^\lambda, \text{ at } T_c, \\ &\chi'(0) \sim \text{const} - (T - T_c)^{-\alpha}, \quad T > T_c, \end{aligned} \tag{10}$$

if $\lambda, -\alpha > 0$, or

$$\begin{aligned} &\chi'(k) \sim -k^{|\lambda|}, \text{ at } T_c, \\ &\chi'(0) \sim -(T - T_c)^{|\alpha|}, \quad T > T_c, \end{aligned} \tag{11}$$

if $\lambda, -\alpha < 0$. Let d be the dimension of the system and $\epsilon \equiv 4 - d$. Using the technique of Ref. 3, we find

$$\lambda = \epsilon\left(\frac{m-2}{m+4}\right) + \epsilon^2 \frac{(m+1)}{(m+4)^2}\left(\frac{13}{2} - \frac{15}{m+4}\right) + O(\epsilon^3). \tag{12}$$

Remember that m is the number of components of the Bose (complex) field. It is analogous to $n/2$ for an n-component *real* field. For large m, we find[4]

$$\lambda = 4 - d - 8(2/d + d - 3)S_d m^{-1} + O(m^{-2}), \quad 2 < d < 4, \tag{13}$$

$$S_d \equiv \frac{\sin\pi(\frac{1}{2}d - 1)}{\pi(\frac{1}{2}d - 1)B(\frac{1}{2}d - 1, \frac{1}{2}d - 1)},$$

where B is the beta function. The exponent α can be obtained from λ via the scaling law[7]

$$-\alpha = \lambda\gamma/(2 - \pi). \tag{14}$$

Equation (12) shows that, for $m = 1$, ϵ small, $\lambda < 0$, so that (11) holds. For $\epsilon = 1$, the ϵ^2 term barely turns λ positive. Thus, the sign of λ is uncertain[8] for $\epsilon = 1$, $m = 1$. For $m \geq 2$, (12) and (13) indicate that $\lambda = 0$ and (10) holds. The expansion for λ and η in ϵ has been given by Wilson[3] and by Nickel.[9] We give the large-m expression here:

$$\gamma = (\tfrac{1}{2}d - 1)^{-1}(1 - 3m^{-1}S_d) + O(m^{-2}), \tag{15}$$

$$\eta = 2(4/d - 1)m^{-1}S_d + O(m^{-2}), \quad 2 < d < 4. \tag{16}$$

III. *Integrated density matrix and effective Hamiltonian.*—The density matrix given by (5) specifies the probability distribution for $a_{\vec{k}}$ and $c_{\vec{k}}$ for all $\vec{k}$. It exhibits the interactions in the microscopic scale clearly, but its implication for large-scale behavior is unclear. Since only field variables of small $\vec{k}$ are of interest for critical behavior, we integrate ρ over all those $a_{\vec{k}}$ and $c_{\vec{k}}$ with $k > \Lambda$, where Λ^{-1} is much larger than the microscopic size, but still small compared to the correlation length ξ. The integrated density matrix ρ', which defines a new Hamiltonian $\mathcal{H}'$,

$$\rho' \propto \exp(-\mathcal{H}'), \tag{17}$$

is exactly equivalent to ρ as far as $a_{\vec{k}}, c_{\vec{k}}$ with $k < \Lambda$ are concerned. $\mathcal{H}'$ contains only these variables, but its appearance is very different from $\mathcal{H}$. In the diagram language, $\mathcal{H}'$ contains terms shown in Fig. 1. The shaded areas are effective coupling constants. They represent all diagrams with the specified external lines of momenta less than Λ, but with *all internal lines having momenta larger than* Λ. The last fact guarantees that these coupling constants are nonsingular at T_c for external momenta much less than Λ and can be expanded in powers of the external momenta. Furthermore, s' (the second diagram in Fig. 1, which is a self-energy term for the Coulomb field), is positive, because it is an average of a

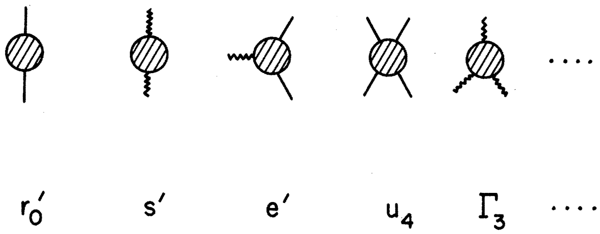

FIG. 1. Effective coupling constants for fields of momenta less than Λ due to fields of momenta greater than Λ. Saw-tooth lines denote the Coulomb field and solid lines, bosons.

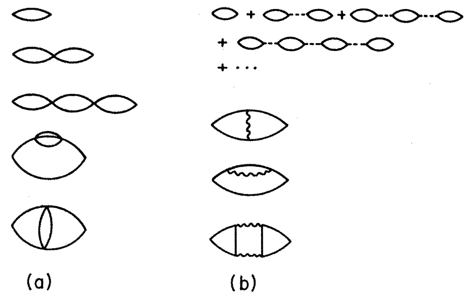

FIG. 2. Diagrams for χ_n for the perturbation calculation. (a) Terms up to $O(\epsilon^2)$. (b) Terms up to $O(1/m)$.

positive quantity (i.e., $e^2\rho_{\vec{k}}\rho_{\vec{k}}^*$ with $a_{\vec{q}}, c_{\vec{q}}$, $q<\Lambda$ excluded in the definition of $\rho_{\vec{k}}$ and in averaging). We write, following Fig. 1,

$$\mathcal{H}' = \sum_{k<\Lambda}\Big[(r_0'+k^2)\sum_{j=1}^{m} a_{\vec{k}}^* a_{j\vec{k}} + (s'+k^2)c_{\vec{k}}^* c_{\vec{k}} + \frac{e'}{\sqrt{2}}\rho_{\vec{k}}'(c_{\vec{k}}^* + c_{-\vec{k}}) + \tfrac{1}{2}u_4\rho_{\vec{k}}'\rho_{-\vec{k}'} + \sum_{k'<\Lambda}\Gamma_3 c_{\vec{k}}^* c_{\vec{k}'}^* c_{\vec{k}+\vec{k}'} + \cdots\Big], \tag{18}$$

where

$$\rho_{\vec{k}}' \equiv \sum_{j=1}^{m}\sum_{p,|\vec{p}+\vec{k}|<\Lambda} a_{j\vec{p}}^* a_{j\vec{p}+\vec{k}}. \tag{19}$$

The reason for treating the Coulomb interaction as a separate field should now be clear: Otherwise, the new coupling constants would still be singular.

Even though $\mathcal{H}'$ looks far more complicated than $\mathcal{H}$, it tells qualitatively the behavior averaged over a scale Λ^{-1}. In particular, the appearance of s' removes the $1/k^2$ Coulomb singularity, and there is no longer any long-range force. *The model defined by $\mathcal{H}'$ is just another model for a neutral Bose system, provided that the restriction (3) is removed.* Now we invoke the universality hypothesis and conclude that the critical behaviors implied by $\mathcal{H}'$ with the restriction (3) removed are those of a neutral Bose system. The correlation functions so calculated (denoted by χ_n and G_n) are of course not quite the same as χ and G, but the connections among them are not hard to establish.

The only difference between G_n and G is that G_n includes a term proportional to $(e'^2/s')\times$(boson density) (i.e., a "Hartree" term) in its self-energy, but G does not because of the restriction (3). This term can be absorbed as a correction to r_0' and will affect the transition temperature but not the exponents. Thus, the exponents η and γ are the same as those in the neutral case since they are directly obtained from G.

The connections among the density correlation functions are less trival. Note that $\rho_{\vec{k}}'$ is not the same as $\rho_{\vec{k}}$. Equation (19) shows that $\rho_{\vec{k}}'$ involves two boson lines of momenta less than Λ. Thus, any diagram for χ_n can be separated into two pieces if we cut two such boson lines. We now write the Coulomb-field self-energy as

$$e^2\chi' = s'' + \Pi, \tag{20}$$

where s'' (including s') is the sum of all self-energy diagrams which cannot be separated by cutting two boson lines of momentum less than Λ, and Π is the rest. We then sum all corrections to the boson-Coulomb-field vertex which cannot be separated by cutting two boson lines. Let this sum plus e' be e''. Simple counting shows that

$$e''^2\chi_n = \Pi - \Pi D \Pi, \tag{21}$$

$$e^2\chi = s'' + \Pi - s''D\Pi - \Pi D s'' - \Pi D \Pi = k^2(s''+\Pi)D. \tag{22}$$

Eliminating Π, D from (7), (21)–(23), we find the desired expressions for χ, χ' in terms of χ_n:

$$e^2\chi' = s''[1-(e''^2/s'')\chi_n]^{-1}, \tag{23}$$

$$e^2\chi = k^2\left[1+\frac{k^2}{s''}\left(1-\frac{e''^2}{s''}\chi_n\right)\right]^{-1}. \tag{24}$$

For our purpose, it is sufficient to know that e'' and s'' are finite constants at zero momenta and at T_c.

IV. *Calculations.*—To $O(\epsilon^2)$, (8) and (12) can be verified by evaluating the diagrams shown in Fig. 2(a), using techniques of Ref. 3. The details are given elsewhere.[10] The $O(1/m)$ calculation goes as follows. For large m diagrams with more closed loops will be more important. Taking the effective coupling constant u to be of $O(1/m)$, correlation functions can be computed order by order in $1/m$ in a similar fashion as the ϵ expansion. To see the loops explicitly we express u as a dashed line in Fig. 2(b). The first diagram in Fig. 2(b) is just a geometric series:

$$m\Pi_0(1+mu\Pi_0)^{-1}. \tag{25}$$

For small k, $\Pi_0(k)\sim k^{d-4}$, which blows up as $k\to 0$. Equation (25) can be expanded as

$$m[(mu)^{-1}-(mu)^{-2}\Pi_0^{-1}+(mu)^{-3}\Pi_0^{-2}+\cdots]. \tag{26}$$

Thus, the first two terms have already the form

$$\text{const}+\text{const}'k^{\lambda}, \tag{27}$$

with $\lambda=4-d$. To compute the next order term, we need

$$\Pi'(1+mu\Pi_0)^{-2}=(mu)^{-2}\Pi_0^{-2}\Pi'+\cdots, \tag{28}$$

where Π' is given by the rest of Fig. 2(b), where each wavy line is a factor $u(1+mu\Pi_0)^{-1}$. Apart from an overall factor, the contributions of the diagrams in Fig. 2(b) are, respectively,

$$\chi_n\sim\text{const}-k^{4-d}+4m^{-1}S_dk^{4-d}(\ln k+\text{const})+4(4/d-1)m^{-1}S_dk^{4-d}(\ln k+\text{const}) \\ +8(d-3)m^{-1}S_dk^{4-d}(\ln k+\text{const})+\text{const}, \tag{29}$$

which leads to (13). The calculation of λ and η is similar and will be described elsewhere.[10]

With the information we now have for the neutral system, we can obtain χ' and χ for the charged system via (23) and (24). *Case 1.* If $\lambda>0$ [and thus $-\alpha>0$ by (14)] $\chi_n(0)$ is a finite constant at T_c. Without divergence, we can choose a small enough Λ to make χ_n small enough so that the denominator of (23) is positive and (10) follows. *Case 2.* If $\lambda, -\alpha$ are negative, $\chi_n(0)\to\infty$ at T_c, and (11) follows from (23). Since our results indicate that λ is unlikely to be less than -2, (9) should always be valid.

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