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Particle Motion in Magnetic Mirrors with High-Frequency Electric Field Fluctuations

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Particle motion in a magnetic mirror containing high-frequency short-wavelength longitudinal electric fields is followed numerically for many mirror bounce periods. It is found that while the magnetic moment undergoes large rapid changes, these fluctuations seldom imply rapid particle loss from the mirror. This behavior is shown to be completely consistent with nonlinear superadiabatic theory, wherein containment properties of the trap are affected only when particles cross isolated singular surfaces in phase space.

Microinstabilities for warm plasma contained by magnetic mirrors are typically characterized by longitudinal electric fields having frequencies near multiples of the ion cyclotron frequency, and wavelengths, in units of the ion gyroradius, long along and short across the external $\vec{B}$ field. The all-important question of particle containment in such fields is a difficult one and must be attacked by both numerical and analytic methods. In this Letter we report the results of a study where, for a small set of initial conditions, we numerically follow particle motions for many mirror cycles under various field conditions. These results are shown to be consistent with an approximate averaged analytic theory. Also, the details of the numerical results have enabled us to determine the relative importance of various aspects of the theory.

The basis of the approximate analytic theory is that, while the magnetic moment is no longer an adiabatic invariant in such fluctuations, an invariant still exists almost everywhere along the particle's trajectory.[1] According to this "superadiabatic" theory, it is only when a particle is forced to cross an isolated singular surface (separatrix) in phase space that this invariant is broken and a nonperiodic change induced in the particle's mirror motion. Containment of a particle, once contained in this field configuration, is then determined by the accumulative sum of these nonadiabatic transitions along the entire path history of the particle.

The field configuration used in the simulation has been chosen to be a divergence- and curl-free weak magnetic mirror field with components $B_Z(X,Z)$ and $B_X(X,Z)$, where $B_Z \gg B_X$, and an electric potential $\varphi(Z,X,t)$ with frequency chosen to be near the minimum of the second harmonic of the cyclotron frequency. In these functionally fixed fields, the exact equations of motion are numerically solved using a standard simulation numerical algorithm with a corrector that renders the orbit exact if all fields were independent of X, Z, t.[2] With these techniques, particle trajectories with differing initial phases have been calculated for times up to $10^5/2\pi$ cyclotron periods, which for D_2 in a 10-kG field corresponds to 2 msec. For the same set of initial particle phases the electric field configuration is then changed by independently varying the frequency, amplitude, and wavelengths both perpendicular and parallel to $\vec{B}$. Typical phase-space projections for a number of different conditions are presented in Figs. 1 through 4. The time units are basic cyclotron times $\Omega_0 t$ in all plots, extending from 0 to 10^4. In all examples the magnetic field has a mirror ratio of 1.1 and a mirror maximum at $Z = \pm L_M$ ($L_M = 140$ gyroradii), and the unperturbed particle is mirroring at $Z = \pm L_M/2$ with a full Z bounce period $\Omega_0 t \approx 2 \times 10^3$.

In order that the important features of these trajectories become apparent it is first necessary to review the predictions of the superadiabatic theory.[1] First of all, in the absence of any Z variation of φ and $\vec{B}$, the theory predicts that the electric field causes periodic variation of the magnetic moment μ at a frequency $\bar{\omega}$, which for the qualitative purposes of this paper can be characterized by the functional form

$$\bar{\omega} = [(\omega - l\Omega)^2 - (e\varphi k^2/2m\Omega)^2 G_{l,k}(H)]^{1/2}, \qquad (1)$$

where ω, k, and φ are the frequency, wave vector perpendicular to $\vec{B}$, and amplitude of the electric potential; l is the harmonic number which has been assumed close to ω/Ω, and $G_{l,k}(H)$ is a complicated function of the time-averaged perpendicular Hamiltonian, H, having for $l = 2$ an upper bound equal to unity. The essence of the superadiabatic theory is that if the rate of spatial variation induced by mirror fields $\vec{B}(Z,X)$ or nonflute behavior ($\partial\varphi/\partial Z \neq 0$) of the potential is slow compared to the frequency $\bar{\omega}$, then the quasiperiodic motion implies the existence of an adiabatic invariant J which replaces the usual magnetic moment, and neglecting slow drift motion, implies cyclic motion in $Z(t)$. However, in general, because of the Z dependence of $\omega - l\Omega$, and φ, as a particle traverses the mirror, it may be forced to cross a separatrix where $\bar{\omega} = 0$, and here a typically small nonperiodic transition takes place in J. This change in turn implies a difference in the Z motion, with possible particle loss out the end of the mirror. The problem then is understanding the various ways this nonadiabatic transition can take place, and its implication on confinement.

Two classifications are immediately apparent from (1); one is where $\omega - l\Omega \neq 0$ anywhere in the mirror (nonresonant) and the resonant case where

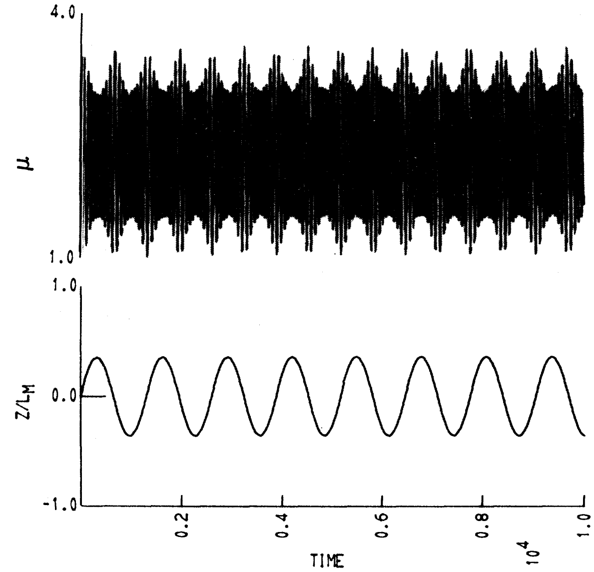

FIG. 1. Magnetic moment, and particle motion along mirror field when electric field is nonresonant, flute, with supercritical amplitude $e\varphi(mv_0^2/2) = 0.1$.

$\omega = l\Omega$ somewhere along the mirroring orbit.

In the first, nonresonant case, we see in order that $\bar{\omega} = 0$, φ must exceed a critical value φ_c given (for $l = 2$) by

$$\frac{e\varphi_c}{m\Omega^2/k^2} = \frac{2(\omega - l\Omega)_{\min}}{\Omega}. \qquad (2)$$

For potential fluctuations above this level, particles still may not cross a separatrix, so that all that can be said in general is that for $\varphi < \varphi_c$, particles once contained should be contained for long times, and for $\varphi > \varphi_c$ the motion may begin to scatter.

For such nonresonant cases Fig. 1 illustrates typical computer results when $\varphi > \varphi_c$, where φ has been chosen flutelike (independent of Z). The particular case illustrated corresponds to $\delta\omega/\Omega_{\min} = 0.05$, and a potential amplitude $e\varphi = 2e\varphi_c = 10\%$ of the initial kinetic energy, indeed a huge fraction of the kinetic energy. The μ-versus-time plot shows a large change in the average value $\bar{\mu}$, as well as a large-amplitude fluctuation at a frequency $\omega - l\Omega(Z)$. Approximately one half of the total number of particles obtain a decreased $\bar{\mu}$ and for sufficiently large φ are lost in one mirror transit time, which is entirely consistent with the theory. All particles surviving loss on the first mirror transit behave similar to the example shown with an extremely regular Z bounce motion and are contained for times $\Omega t > 10^5$ (shown in Fig. 1 only up

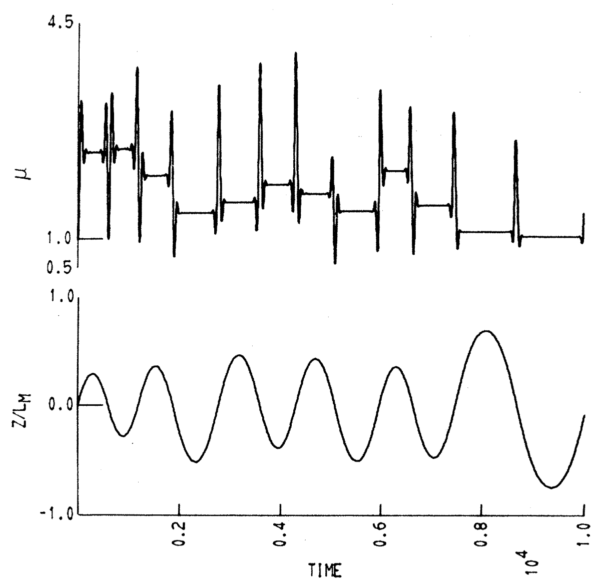

FIG. 2. Magnetic moment, and particle motion along mirror field when electric field is nonresonant, nonflute, with just critical amplitude, $e\varphi(mv_0^2/2)=0.2$.

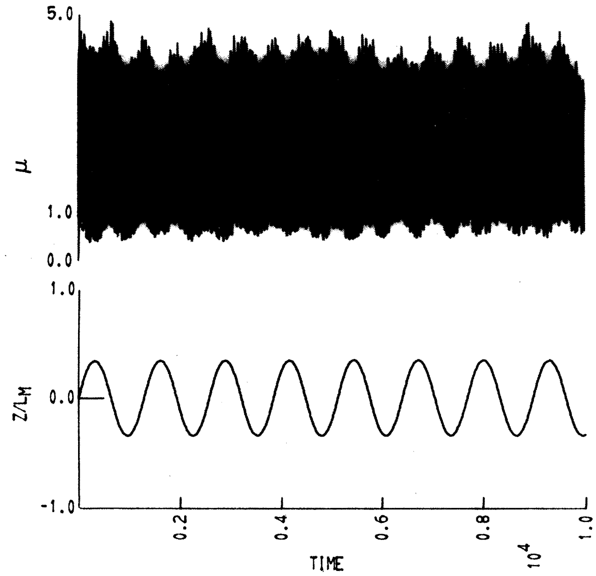

FIG. 3. Magnetic moment, and particle motion along mirror field when electric field is resonant, flute, and extremely large amplitude, $e\varphi/(mv_0^2/2)=0.5$.

to times $\Omega t = 10^4$). When $\varphi > \varphi_c$ for any choice of initial phase there is observed at most only weak scatter in the mirroring (none is observed when $\varphi < \varphi_c$). This is evidence that for the moderate-φ nonresonant flute case most particles are not forced into orbits that cross separatrices.

For the nonresonant strongly nonflute case when $\varphi = \varphi_c \exp(-Z^2/L_\varphi^2)$, $L_\varphi/L_M = 0.125$, Fig. 2 is representative, showing strong scatterings in the mirroring position. For Fig. 2, φ is extremely small in the mirror where the particle is turning around, in which case the adiabatic invariant J becomes approximately μ. Therefore, the nonadiabatic changes in J are, in fact, approximately illustrated in Fig. 2(a) by the differing values of μ where Z is a maximum or minimum. Also, when the Z variation of φ was decreased to $L_\varphi/L_M = 0.25$, one half that of Fig. 2, then variations in J appeared to be almost negligible. This is evidence that the changes in J have an exponential dependence on L_φ/L_M, and this is consistent with the expected changes in behavior due to the variation of the small parameter of adiabatic theory, as long as the particle does not cross a separatrix.[3]

When φ was increased to much larger values and held flutelike the computer results show containment of all particles up to times $\Omega_0 t = 10^5$. This behavior occurs in both the nonresonant and resonant cases and results look typically as those in Fig. 3 which itself is a resonant case where $\omega - l\Omega = 0$ near the mirror midpoint. Again the plot is shown only up to times $\Omega_0 t = 10^4$. In this strong-field limit it appears that the linear frequency $\omega - l\Omega$ is almost irrelevant; and in this limit the adiabatic invariant becomes only a function of Z through the $\varphi(Z)$ amplitude variation.[1] As φ was always held constant in these large-φ limits, J becomes an exact constant of the motion; and particles are perfectly contained. However, in the vicinity of singular points in phase space the coefficients of φ in the equations of motion identically vanish, and then near these points the linear orbit correction, $\omega - l\Omega$, is important. Therefore, in all strong-φ cases, according to superadiabatic theory, and as evidenced in all computer runs, only very infrequent scattering should occur, containment being almost as good as when $\varphi < \varphi_c$ in the nonresonant cases. In a sense containment is observed to be even better for the strong-φ case, since here most of the particles are contained for long times, whereas in the nonresonant moderate-φ case a large fraction of the particles are lost on the first mirror transit. Another way of understanding this particular phenomenon is that the strong nonlinearities rapidly detune any phase-coherent resonance, and thereby limit secular-type excursions. Such a nonlinear saturation of particle losses may be the explanation of reported experimental improvement in containment in the presence of external rf sources on a

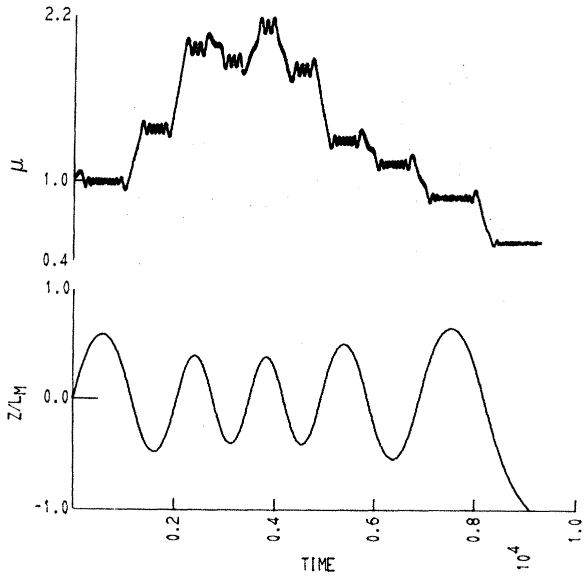

FIG. 4. Magnetic moment, and particle motion along mirror field when electric field is resonant, flute, and moderate amplitude, $e\varphi/(mv_0^2/2)=0.01$.

mirror machine.[4]

For low and intermediate values of φ, resonant modes act on the particles quite differently than in the nonresonant case. Where $\omega - l\Omega = 0$, the system is strongly nonlinear, and near the ends of the system where the nonlinear terms in (1) are small compared to $\omega - l\Omega$, the system is essentially linear. For such an implied change in phase-space topology the particle is almost always forced to cross singular surfaces during one mirror transit and thereby scatter appreciably in a number of transit times. Figure 4 illustrates this well as the field configuration and initial particle phases are identical to those of Fig. 3, except that in Fig. 4 the potential φ is 50 times *smaller*. Typically, the cumulative changes in μ are sufficient to cause particle loss at time $\Omega t < 10^4$.

In summary, what these many computer runs have shown is that the containment properties of weak mirrors seem to be completely consistent with those calculated with the superadiabatic theory at least for times of order $10^5/2\pi$ cyclotron periods. Furthermore, the overall results are that containment is a strong function of orbit phase, and if almost everywhere along the trajectory the conditions $\Omega_0 \gg \bar{\omega} \gg \omega_{\text{bounce}}$ are satisfied, then *moderate* values or a strong Z variation of φ lead to the most rapid particle losses.

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Recurrence of Nonlinear Ion Acoustic Waves

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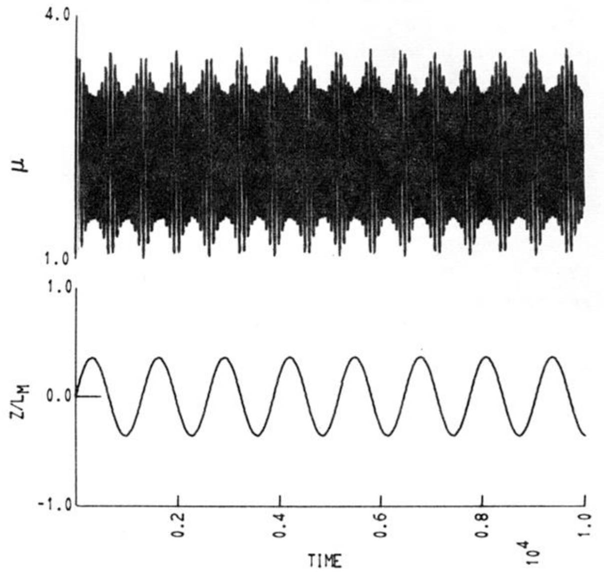
The recurrence time of ion acoustic waves in a plasma has been obtained, as a function of the parameters describing the plasma, on the basis of the Korteweg–de Vries (KdV) equation and Whitham's extended KdV equation. Direct numerical solutions of the ionic Vlasov equation have also been obtained which quantitatively confirm the predictions of the KdV equation.

Recurrence of the initial state in nonlinear wave propagation (implying an absence of relaxation to a state of energy equipartition) was first observed by Fermi, Pasta, and Ulam[1] in numerical experiments on nonlinear lattices, and was subsequently explained in terms of solitons and the Korteweg-de Vries (KdV) equation by Zabusky and Kruskal.[2] Recent experimental observations[3,4] of recurrence and KdV-type behavior of signals propagating on nonlinear transmission lines have led to renewed interest in this phenomenon. We shall report on a theoretical-numerical analysis of recurrence of ion acoustic waves in plasmas in a regime where solitons have been observed[5]



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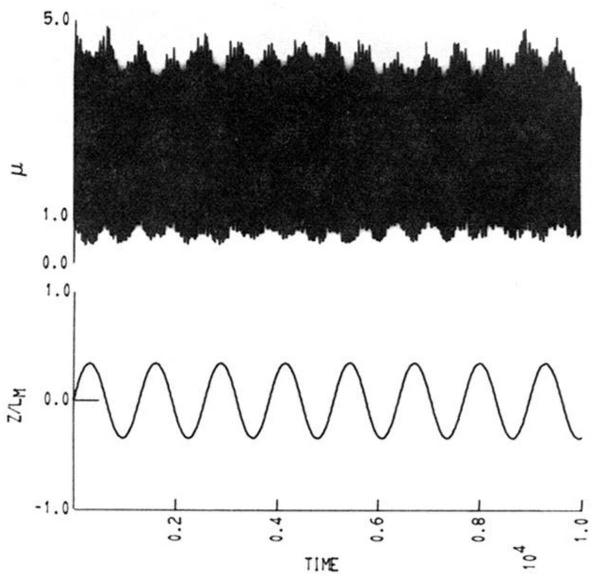

FIG. 3. Magnetic moment, and particle motion along mirror field when electric field is resonant, flute, and extremely large amplitude, $e\varphi/(mv_0^2/2) = 0.5$.