tion. Clustering also occurs for potassium ions in nitrogen and will introduce further corrections.

Since the systematic errors in Elford's data are of the same magnitude as the apparent pressure dependence and show the same type of variation, it must be concluded that his results do not raise serious doubts as to the validity of the transport theory.

I would like to thank Dr. E. W. McDaniel and Dr. G. M. Thomson for their stimulating discussions.

*Research sponsored by the U. S. Air Force Office of Scientific Research, Office of Aerospace Research, under Grant No. AFOSR-71-2097.

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New Theoretical Values of the 2^3P Fine-Structure Splittings of Helium

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(Received 13 April 1972)

We report here new theoretical values for the $2\,^3P$ fine-structure splittings in helium. Comparison with experiment provides significant confirmation of the newly computed effects.

Precision measurements of the fine structure of the $2\,^3P$ level of helium make possible a precise test of the theory of electron-electron interaction. Alternatively, assuming the validity of the theory, a high-precision determination of the fine-structure constant[1] can be obtained.

Recent measurements[2] have determined both fine-structure intervals to ~1 ppm. We report here new theoretical values for the splittings. While the theoretical precision achieved is not sufficient to fully exploit the precision of the experimental results, the results do nevertheless provide significant confirmation of the newly computed effects.

The complete evaluation of the splittings to 1 ppm can be divided into four well-defined tasks. (1) One calculates, using Schrödinger wave functions, the expectation values of the usual spin-dependent fine-structure operator[3]

$$H^4 = H_{z,\mathrm{s.o.}}{}^4 + H_{e,\mathrm{s.o.}}{}^4 + H_{e,\mathrm{s.s.}}{}^4,$$

a contribution of order $\alpha^4 mc^2$; $H_{z,\mathrm{s.o.}}{}^4$ is the spin-orbit interaction with the nucleus, $H_{e,\mathrm{s.o.}}{}^4$ is the spin-orbit interaction between electrons, and $H_{e,\mathrm{s.s.}}{}^4$ is the spin-spin interaction between electrons. This calculation has been done by Schwartz[4] and by Schiff, Pekeris, and Lifson[5] to an accuracy of ~1 ppm. To include the $\alpha^5 mc^2$ term, the electron magnetic moment is multiplied by $(1+\alpha/2\pi)$, yielding the operator

$$H^5 = (\alpha/2\pi)[2H_{z,\mathrm{s.o.}}{}^4 + \tfrac{4}{3}H_{e,\mathrm{s.o.}}{}^4 + 2H_{e,\mathrm{s.s.}}{}^4],$$

i.e., multiples of the $\alpha^4 mc^2$ operators. (2) The $\alpha^4 mc^2$ operators are calculated in second order, a contribution of order $\alpha^6 mc^2$. Here the spin-independent operators must be included. (3) A spin-dependent operator of higher order must be derived from a general theory. (4) One calculates the expectation value of this operator, again with Schrödinger wave functions.

The covariant two-particle Bethe-Salpeter equation, with the nucleus treated as a fixed source of Coulomb field, provides the starting point for the general theory. A systematic reduction of

TABLE I. $\alpha^6 mc^2$ operators in atomic units ($m=1$) and their contributions to the splittings. Operators to the left of the asterisk act on the left-hand wave function.

Operator	ν_{01}[MHz]	ν_{12}[MHz]
$H_1^6 = \frac{3}{8}\alpha^3(z\alpha)\,\nabla_1^2 * \frac{1}{r_1^3}\,\vec{\sigma}_1\cdot(\vec{r}_1\times\vec{p}_1)$	2.08639	4.17279
$H_2^6 = -\alpha^3(z\alpha)\,\frac{1}{r^3 r_1^3}\,\vec{\sigma}_1\cdot(\vec{r}_1\times\vec{r})(\vec{r}\cdot\vec{p}_2)$	.15379	.30758
$H_3^6 = \frac{1}{2}\alpha^3(z\alpha)\,\frac{1}{r^3 r_1^3}\,(\vec{\sigma}_1\cdot\vec{r})(\vec{\sigma}_2\cdot\vec{r}_1)$	.09609	-.03843
$H_4^6 = \frac{1}{2}\alpha^4\,\frac{1}{r^4}\,\vec{\sigma}_1\cdot(\vec{r}\times\vec{p}_2)$	.32227	.64454
$H_5^6 = -\frac{1}{2}\alpha^4\,\frac{1}{r^6}\,(\vec{\sigma}_1\cdot\vec{r})(\vec{\sigma}_2\cdot\vec{r})$	.75718	-.30287
$H_6^6 = -\frac{5}{8}\alpha^4\,\nabla_1^2 * \frac{1}{r^3}\,\vec{\sigma}_1\cdot(\vec{r}\times\vec{p}_1)$	-.86830	-1.73661
$H_7^6 = \frac{3}{4}\alpha^4\,\nabla_1^2 * \frac{1}{r^3}\,\vec{\sigma}_1\cdot(\vec{r}\times\vec{p}_2)$	-1.59865	-3.19730
$H_8^6 = i\frac{1}{4}\alpha^4\,\nabla_1^2 * \frac{1}{r}\,\vec{\sigma}_1\cdot(\vec{p}_1\times\vec{p}_2)$	.00168	.00337
$H_9^6 = i\frac{3}{4}\alpha^4\,\nabla_1^2 * \frac{1}{r^3}\,(\vec{r}\cdot\vec{p}_2)\,\vec{\sigma}_1\cdot(\vec{r}\times\vec{p}_1)$	.06027	.12055
$H_{10}^6 = i\frac{3}{8}\alpha^4\,\frac{1}{r^5}\,\vec{\sigma}_1\cdot(\vec{r}\times(\vec{r}\cdot\vec{p}_2)\vec{p}_1)$	-.08109	-.16219
$H_{11}^6 = -\frac{3}{16}\alpha^4\,\frac{1}{r^5}\,\vec{\sigma}_2\cdot(\vec{r}\times(\vec{\sigma}_1\cdot(\vec{r}\times\vec{p}_1))\vec{p}_2)$	-.04139	.01655
$H_{12}^6 = -\frac{1}{16}\alpha^4\,\frac{1}{r^3}\,(\vec{\sigma}_1\cdot\vec{p}_2)(\vec{\sigma}_2\cdot\vec{p}_1)$	-.09664	.03865
$H_{13}^6 = -\frac{3}{2}\alpha^4\,\nabla_1^2 * \frac{1}{r^5}\,(\vec{\sigma}_1\cdot\vec{r})(\vec{\sigma}_2\cdot\vec{r})$	-7.34250	2.93700
$H_{14}^6 = i\frac{1}{4}\alpha^4\,\nabla_1^2 * \frac{1}{r^3}\,(\vec{\sigma}_1\cdot\vec{r})(\vec{\sigma}_2\cdot\vec{p}_1)$	1.26991	-.50796
$H_{15}^6 = -i\frac{1}{8}\alpha^4\,\nabla_1^2 * \frac{1}{r^3}[(\vec{\sigma}_1\cdot\vec{r})(\vec{\sigma}_2\cdot\vec{p}_2) + (\vec{\sigma}_2\cdot\vec{r})(\vec{\sigma}_1\cdot\vec{p}_2) - \frac{3}{r^2}(\vec{\sigma}_1\cdot\vec{r})(\vec{\sigma}_2\cdot\vec{r})(\vec{r}\cdot\vec{p}_2)]$	1.98973	-.79589
$H_{16}^{6\,a} = -.328\left(\frac{\alpha}{\pi}\right)^2\left(2H_{z,so}^4 + \frac{4}{3}H_{e,so}^4\right) + \left(2(-.328)\left(\frac{\alpha}{\pi}\right)^2 + \left(\frac{\alpha}{2\pi}\right)^2\right)H_{e,ss}^4$	-.05432	.02158
$H_N' = +(m/M_{He})\,\alpha(z\alpha)\,\frac{1}{r_1^3}\,\vec{\sigma}_1\cdot(\vec{r}_1\times\vec{p}_2)$	4.78236	9.56472

[a] This operator represents the next-order modification of the electron magnetic moment. These are the only radiative corrections of order $\alpha^6 mc^2$.

this equation to a nonrelativistic form,[6] carried out to the required order, provides both a justification of the separation described above and an explicit form for the operators H^6 of order $\alpha^6 mc^2$. These H^6 operators, displayed in Table I, are to be used only for calculating the ^3P splittings as

TABLE II. Second-order energies.

	ν_{01} (MHz)	ν_{12} (MHz)
^3P	5.006 (0.18)	−0.360 (0.36)
^1P	6.505 (0.06)	−6.505 (0.06)
^3D	0.027 (0.002)	0.072 (0.003)
Sum	11.60 (±0.18)	−6.79 (±0.36)

most of their expectation values converge only with spatially antisymmetric wave functions, and all spin-independent terms have been dropped. It is possible that α^7mc^2 terms contribute at the 1-ppm level, but these have not been evaluated.

We have also derived corrections of order $(m/M_{\mathrm{He}})\alpha^4mc^2$ due to the recoil motion of the nucleus, essentially by analogy with old-fashioned calculations of the α^4mc^2 terms. These motional effects may be taken into account by including the expectation value of

$$H_N = -(m/M_{\mathrm{He}})H_{z,\mathrm{s.o.}}^{\ 4} - 3(m/M_{\mathrm{He}}) \times (H_{e,\mathrm{s.o.}}^{\ 4} + H_{e,\mathrm{s.s.}}^{\ 4}) + H_N'$$

(H_N' is displayed in Table I). Also one finds the well-known mass-polarization operator which contributes in the second-order calculation when mixed with spin-dependent operators.

The contributions in second order of the α^4mc^2 operators (plus mass polarization) have been calculated[7] using the method of Dalgarno and Lewis.[8] One solves variationally an inhomogeneous Schrödinger equation for the perturbation to the wave function; the energy perturbations are then given by integrals. There are eighteen contributions when the symmetry of the perturbation is ^3P, three when it is ^1P, and six when it is ^3D (all odd parity), which affect the large fine-structure interval. A sequence of eight calculations is done with up to 165 Hylleraas terms in the expansion of the wave-function perturbation, and the results carefully extrapolated. Five of the ^3P perturbations were calculated using some additional singular terms in the expansions, reflecting the singularities of the α^4mc^2 operators; this significantly improved the accuracy, i.e., convergence of the eight computed energies. The results are given in Table II. There are also contributions to the small interval when the symmetry of the wave-function perturbation is ^1D or ^3F; these have not been computed. Essentially all of the uncertainty in the calculation of the large interval comes from the second-order energies, and by far the most uncertain contribution comes from the α^4mc^2 operator proportional to the square of the Laplacian mixed with $H_{z,\mathrm{s.o.}}^{\ 4}$, for which the eight energies converged rather poorly.

The expectation values of H^6 and H_N' have also been calculated.[9] To do this a computer program was developed which calculates helium $^3P_0$ expectation values of arbitrary operators with Hylleraas wave functions. All algebraic manipulations are done automatically, the operators being input essentially as written in Table I. This procedure requires integrals over arbitrary powers of the Hylleraas variables. The splittings are derived

TABLE III. Theoretical contributions to fine-structure intervals (MHz). The values of α^{-1}, C, and R_∞ are 137.036 02(21) (1.5 ppm),[a] 2.997 925 0(10) $\times 10^{10}$ cm sec^{-1} (0.3 ppm),[b] 109 737.312 cm^{-1} (0.1 ppm),[b] respectively. Thus $\frac{1}{2}\alpha^2CR_\infty = 87.594\,25 \times 10^3$ MHz (3 ppm).[c]

Interval	α^4mc^2	α^5mc^2	$\frac{m}{M_{\mathrm{He}}}\alpha^4mc^2$	Second order	α^6mc^2	ν_{theo}	ν_{expt}	$\nu_{\mathrm{theo}} - \nu_{\mathrm{expt}}$
ν_{01}	29 564.567[d]	54.708[e]	−10.707[f]	11.60	−3.331	29 616.83	29 616.864[g]	−0.03 = 1 ppm
	±0.006		±0.000 44	±0.18	±0.0039	±0.18	±0.036	±0.18
	(0.21 ppm)		(0.015 ppm)	(6 ppm)	(0.13 ppm)	(6 ppm)	(1.2 ppm)	
ν_{12}	2317.103[d]	−22.548[e]	1.952[f]	−6.79	1.542	2291.36	2291.196[g]	+0.16 = 70 ppm
	±0.0017		±0.000 88	±0.36	±0.0068	±0.36	±0.005	±0.36
	(0.76 ppm)		(0.39 ppm)	(157 ppm)	(3.0 ppm)	(157 ppm)	(2.2 ppm)	

[a] Ref. 1.

[b] B. N. Taylor, W. H. Parker, and D. N. Langenberg, Rev. Mod. Phys. 41, 375 (1969).

[c] Uncertainty in α is not included in quoted errors.

[d] These values are taken from Ref. 4, Eq. (13). R_∞ is used rather than R_{He} since use of R_{He} would include a portion of the nuclear-motion term, which we wish to tabulate separately.

[e] R_∞ is used. The values of C_z, C_e, and D_b are taken from Ref. 4.

[f] R_∞ is used. We take $m/M_{\mathrm{He}} = 1.370\,97 \times 10^{-4}$.

[g] Ref. 2.

from $J = 0$ expectation values using Racah algebra techniques. The splittings from H^6 and H_N' were calculated with a sequence of eight wave functions, the results with the largest wave function (165 terms) being given in Table I. For each wave function the total splittings from H^6 and H_N' were calculated and the four sequences of values extrapolated to give the entries of Table II. The convergence is quite smooth and the accuracy obtained is more than sufficient for the present comparison with experiment.

The separate contributions and totals are listed in Table III. ν_{01} agrees well with experiment and yields a value of α accurate to ± 3 ppm and consistent with Ref. 1. Considerable work remains to find ν_{12} to comparable accuracy (~ 10 ppm). Completion and improvement in accuracy of the second-order calculation is mandatory, and at least an estimate of the $\alpha^7 mc^2$ terms is desirable. Consistency of the two values will provide a nontrivial check of the computations. A complete and independent recalculation of the effects evaluated here would provide increased confidence. It does appear that the helium fine structure will ultimately provide a value of α accurate to better than one part per million.

We are grateful to Professor Charles Schwartz for numerous valuable suggestions and his long-time interest in this work.

*Work supported in part by the U. S. Air Force Office of Scientific Research.

†Work supported in part by the U. S. Atomic Energy Commission.

‡Work supported in part by the National Science Foundation.

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Higher-Order Vacuum Polarization Corrections in Muonic Atoms*

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In view of the recently reported discrepancies between theoretical calculations and accurate experimental measurements of muonic x-ray energies for several elements in the energy range 150–440 keV, we have recalculated the vacuum polarization corrections. Our results reduce, but do not eliminate, the discrepancies. We discuss the effects of a possible anomalous coupling of the muon to a speculative scalar meson of the type suggested by Weinberg.

In a recent paper on the experimental test of the theory of muonic atoms, Dixit *et al.*[1] have reported measurements of muonic x-ray energies in a number of elements in the energy range 150–440 keV. They picked for measurement only the higher transitions in atoms with spherical nuclei and made special efforts to measure these transitions with an absolute precision of 15–21 eV. Since these high transitions are only a little affected by the nucleus, the measurements could serve to test the extent to which the existing theory of such hydrogenlike atoms, namely that based on the Dirac equation together with applicable atomic and vacuum-polarization corrections, works. The prinicpal correction to the Dirac energy in this energy range turns out to be that due to e^--e^+ vacuum polarization. These corrections were calculated by Fricke[2]; and, using these calculated numbers, Dixit *et al.*[1] arrived at their theoretical values for the transition energies. Corrections due to nuclear polarization, the screening effect due to electrons, the Lamb shift and relativistic corrections to the reduced mass, etc. are small but were nevertheless taken into