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Axial-Vector Anomalies and the Scaling Property of Field Theory*

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A proof of the Adler-Bardeen theorem is given with the aid of the Callan-Symanzik equation.

The recent realization that the processes $\gamma \to 3\pi$ and $2\gamma \to 3\pi$ will provide basic information[1] about the partially conserved-axial-vector-current (PCAC) triangle anomaly[2] has provoked new interest in this subject. If the anomaly is to provide a test of the relevance of the renormalized perturbation series to hadron physics, it is clearly essential that the value of the anomaly remains the same up to any finite order in perturbation theory.[3] Let us elaborate. Consider a perturbative calculation of the amplitude

$$R_{D\mu\nu}(k,q) = i\int d^4x\, d^4y\, e^{i(kx+qy)}\langle 0|\,T\,\partial A(0)\,V_\mu{}^*(x)\,V_\nu{}^*(y)\,|0\rangle = \epsilon_{\mu\nu\lambda\sigma}k^\lambda q^\sigma f(k^2,q^2,kq) \tag{1}$$

in any renormalizable quantum field theory with fermions and a partially conserved axial-vector current. This theory may be, for example, quantum electrodynamics, or the σ model, or a quark-gluon model. [In what follows the discussion is given for quantum electrodynamics (QED). It is straightforward to modify the discussion to a form appropriate for other theories.] The notation $V_\mu{}^*$ indicates that when calculating $R_{D\mu\nu}$, we omit those diagrams in which the vector current $V_\mu = \bar{\psi}_0\gamma_\mu\psi_0$ hooks eventually onto a photon propagator.[4]

The theorem alluded to above then states the following: To any finite order in perturbation theory, $f(0,0,0)$ is given by the basic fermion triangle graph. This is an extraordinary assertion for it tells us that PCAC and gauge invariance imply the existence of a spectacular cancelation among the infinite[5] collection of Feynman diagrams, thus providing a unique opportunity to confront renormalized perturbation theory with data.[6] Moreover, this theorem provides a springboard for several other deductions on the behavior of field theories.[7]

A constructive proof of this important theorem was given by Adler and Bardeen.[8,9] In this paper we propose an alternative proof. This paper also serves to illustrate the Callan-Symanzik[10] equation at work. So let us begin by writing down the version of this equation appropriate for current correlation functions:

$$\left[\lambda(\alpha)m\frac{\partial}{\partial m}+\beta(\alpha)\frac{\partial}{\partial\alpha}\right]R_{D\mu\nu}(k,q) = -\tfrac{1}{2}R_{SD\mu\nu}(0,k,q)+R_{D\mu\nu}(k,q). \tag{2}$$

Here

$$R_{SD\mu\nu}(p,k,q) \equiv i\int d^4z\, d^4x\, d^4y\, e^{i(pz+kx+qy)}\langle 0|\,TS(z)\,\partial A(0)\,V_\mu{}^*(x)\,V_\nu{}^*(y)\,|0\rangle \tag{3}$$

and $S = 2im_0\bar{\psi}_0\psi_0$. (All unrenormalized quantities are denoted with the subscript 0.)

We now sketch the derivation of Eq. (2). Let $R_{D\mu\nu}{}^0$ and $R_{SD\mu\nu}{}^0$ be the unrenormalized counterparts of $R_{D\mu\nu}$ and $R_{SD\mu\nu}$. Then by definition

$$m_0(\partial/\partial m_0)R_{D\mu\nu}{}^0(k,q) = \tfrac{1}{2}R_{SD\mu\nu}{}^0(0,k,q) + R_{D\mu\nu}{}^0(k,q). \tag{4}$$

The partial differentiation in Eq. (4) is performed with the bare coupling constant the cutoff held fixed. The term $R_{D\mu\nu}{}^0$ appears in Eq. (4) since ∂A explicitly depends on m_0. We now recall that the operators ∂A, S, and $V_\mu{}^*$ are in fact cutoff independent, thanks to the PCAC and conserved-vector-current Ward identities.[4,11] Thus $R_{D\mu\nu}{}^0(k,q) = R_{D\mu\nu}(k,q)$ and $R_{SD\mu\nu}{}^0(p,k,q) = R_{SD\mu\nu}(p,k,q)$. Introducing the definitions $\lambda(\alpha) \equiv m_0/m(\partial m/\partial m_0)$ and $\beta(\alpha) \equiv m_0\,\partial\alpha/\partial m_0$, we obtain Eq. (2). $\lambda(\alpha)$ and $\beta(\alpha)$ are clearly cutoff independent and depend only on α.

We now observe that $R_{SD\mu\nu}$ satisfies a Ward identity:

$$i(k+q)^\lambda R_{S\lambda\mu\nu}(0,k,q) = R_{SD\mu\nu}(0,k,q) - 2R_{D\mu\nu}(k,q), \tag{5}$$

where

$$R_{S\lambda\mu\nu}(p,k,q) = i\int d^4z\, d^4x\, d^4y\, e^{i(pz+kx+qy)}\langle 0|\,TS(z)A_\lambda(0)\,V_\mu{}^*(x)\,V_\nu{}^*(y)\,|0\rangle. \tag{6}$$

We next argue that the Ward identity in Eq. (5) is free from anomalies. As a consequence of crossing symmetry and gauge invariance, any anomalous term in Eq. (5) must have the form $a\epsilon_{\mu\nu\lambda\sigma}k^\lambda q^\sigma$. Since absorptive parts satisfy normal Ward identities,[2] a is a polynomial in the momenta. Weinberg's theorem[12] then shows that $a=0$ to any finite order in renormalized perturbation theory.

One is happy to note that the same expression appears on the right-hand side of Eqs. (2) and (6). This enables the proof to proceed as follows. Expand Eq. (2) in powers of momenta. Such an expansion certainly exists with some nonzero radius of convergence since our fermion mass m does not vanish. Crossing symmetry implies that $R_{S\lambda\mu\nu}(0,k,q) = \epsilon_{\mu\nu\lambda\sigma}(k-q)^\sigma A(\alpha)$ + terms higher order in momenta as $k, q \to 0$. Gauge invariance, however, forces $A(\alpha)$ to vanish. The Ward identity in Eq. (5) now tells us that in the momentum expansion

$$R_{SD\mu\nu}(0,k,q) - 2R_{D\mu\nu}(k,q) = \epsilon_{\mu\nu\lambda\sigma}k^\lambda q^\sigma B(\alpha) + \cdots, \tag{7}$$

the coefficient of expansion $B(\alpha) = A(\alpha) = 0$.[13] Simple dimension counting shows that $f(0,0,0)$ is independent of m. Referring to Eq. (2) we learn that

$$\beta(\alpha)(d/d\alpha)f(0,0,0) = -\tfrac{1}{2}B(\alpha) = 0. \tag{8}$$

Hence $f(0,0,0)$ is independent of α. This completes the proof.[14]

The proof just given holds for theories with only one nonvanishing mass, such as QED. It is easily generalized, however, to theories with several nonvanishing masses, the reason being that the number of Callan-Symanzik equations increases with the complexity of the theory. For example,[15] in massive vector-boson theory with a fermion mass m and a vector-boson mass μ, we would have two equations of the type

$$\lambda\frac{\partial c}{\partial z} + \beta\frac{\partial c}{\partial g} = 0, \tag{9a}$$

$$\lambda'\frac{\partial c}{\partial z} + \beta'\frac{\partial c}{\partial g} = 0. \tag{9b}$$

Here $z \equiv \mu^2/m^2$, and $c(z,q)$ is defined by the expansion $R_{D\mu\nu} \to \epsilon_{\mu\nu\lambda\sigma}k^\lambda q^\sigma c(\mu^2/m^2, g)$ as $k, q \to 0$. $\lambda, \lambda', \beta, \beta'$ are functions of μ^2/m^2 as well as g. Clearly, we reach the same conclusion as before, namely that $c(\mu^2/m^2, g)$ does not depend on g. It then follows that $c(\mu^2/m^2, g)$ does not depend on μ^2/m^2 either.

In applications[16] so far, the right-hand side of the Callan-Symanzik equation is usually eliminated by appealing to Weinberg's theorem and going into the deep Euclidean region. In this paper, we observe that the right-hand side of the Callan-Symanzik equation for $R_{D\mu\nu}$ [Eq. (2)] appears also in a PCAC Ward identity [Eq. (6)]. This information turns out to suffice for our purposes. We note here that this feature is quite general.

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[4]In other words, we keep only one-particle–irreducible diagrams. The standard vector Ward identity $i(p'-p)^\mu\Gamma_\mu{}^0(p,p')=S_F{}^{-1}(p)-S_F{}^{-1}(p')$ does not imply that $Z_2\Gamma_\mu{}^0$ is cutoff independent. Rather, we learn from the decomposition $\Gamma_\mu{}^0=\Gamma_\mu{}^*+[g_{\mu\nu}(p-p')^2-(p-p')_\mu \times(p-p')_\nu]F^\nu(p,p')$ that $Z_2\Gamma_\mu{}^*$ is cutoff independent, but Z_2F_μ may be cutoff dependent. $V_\mu{}^*$ corresponds to $\Gamma_\mu{}^*$. ($\Gamma_\mu{}^0$, S_F, and Z_2 denote the unrenormalized photon vertex part, the unrenormalized fermion propagator, and the fermion wave-function renormalization factor, respectively.)

[5]These statements are all predicated on the belief that any amplitude is completely described by the infinite collection of Feynman diagrams given in the Dyson series. At issue is the question whether or not nonperturbative effects are automatically included.

[6]There are two issues. Firstly, is the PCAC anomaly relevant to hadron phenomena? Secondly, if so, can the value of the PCAC anomaly be calculated in renormalized perturbation theory? See, for example, Aviv and Zee, Ref. 1.

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[14]At the Gell-Mann–Low eigenvalue $\beta(\alpha_{GML})=0$. However, one may still conclude $dc(\alpha)/d\alpha=0$ at $\alpha=\alpha_{GML}$ in renormalized perturbation theory, since up to any finite order $\beta(\alpha)$ is a polynomial.

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