Resolution of the SU(3) Puzzle: $\Gamma(\eta\to\gamma\gamma)/\Gamma(\pi^0\to\gamma\gamma)$*

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Using the Duffin-Kemmer-Petiau (DKP) description of mesons, we are able to render simple SU(3) theory consistent with the supposedly "large" experimental value of $\Gamma(\eta\to\gamma\gamma)/\Gamma(\pi^0\to\gamma\gamma)=129\pm 33$. After discussing η-η' mixing we make a prediction [uncertain because of large errors in $\Gamma_{\rm tot}^{\rm expt}(\pi^0)$ and $\Gamma_{\rm tot}^{\rm expt}(\eta)$] of $\Gamma^{\rm DKP}(\eta'\to\gamma\gamma)=0.32_{-0.32}^{+0.99}$ keV. This is at least 1–2 orders of magnitude lower than the Klein-Gordon (KG) prediction, $\Gamma^{\rm KG}(\eta'\to\gamma\gamma)=52\pm 24$ keV, which in fact is almost ruled out by upper limits set by recent experiments.

Recently we have proposed[1,2] that the Duffin[3]-Kemmer[4]-Petiau[5] (DKP) description of pseudoscalar mesons (P) is better than the conventional Klein-Gordon (KG) description, in situations where there is symmetry breaking.[1,2] Among other things, we were able to explain, simply and numerically, puzzling experimental results in K_{l3} decays[1] (such as the large negative value of the symmetry-breaking parameter ξ, and the discrepancy between the values of the Cabibbo angle obtained from K_{e3} and nuclear $0^+\to 0^+$ β decays), and to make new experimental predictions [such as a dynamically plausible zero in the scalar form factor $f_0(t)$, and a slightly different branching ratio for π_{e3} decay].

In this paper we look at a completely different interaction (electromagnetism) of pseudoscalar mesons. By using the DKP formalism we will first resolve the disagreement between experiment and the simple SU(3) prediction for the decay width ratio $\Gamma(\eta\to\gamma\gamma)/\Gamma(\pi^0\to\gamma\gamma)$. Then after discussing the η-η' mixing angle we will be able to make a testable prediction for the value of $\Gamma(\eta'\to\gamma\gamma)$. This DKP prediction is at least 1–2 orders of magnitude lower than the (almost experimentally ruled out) KG prediction.

(1) *The ratio $\Gamma(\eta\to\gamma\gamma)/\Gamma(\pi^0\to\gamma\gamma)$ in the absence of η-η' mixing.*—We start by considering the ordinary KG description of $\pi^0\to\gamma\gamma$. The matrix element $\mathfrak{M}$ obtained from the Lagrangian density $\mathcal{L}$ is [effectively $\pi^0\vec{E}\cdot\vec{B}$ coupling]

$$\mathcal{L}^{\rm KG}(\pi^0\to\gamma\gamma)=F_\pi\varphi(x)\epsilon_{\mu\nu\lambda\alpha}\partial_\mu A_\nu^{(1)}(x)\partial_\lambda A_\alpha^{(2)}(x), \tag{1}$$

$$\mathfrak{M}^{\rm KG}=-F_\pi(8p_0k_0^{(1)}k_0^{(2)}V^3)^{-1/2}\epsilon_{\mu\nu\lambda\alpha}k_\mu^{(1)}\epsilon_\nu^{(1)}k_\lambda^{(2)}\epsilon_\alpha^{(2)}\exp[ix\cdot(p-k^{(1)}-k^{(2)})], \tag{2}$$

where[6] m_π and p are the mass and four-momentum of the pion, and $\epsilon^{(i)}$ and $k^{(i)}$ are the polarization vectors and four-momenta of the photons. V (T) will be the normalization volume (time). Standard techniques then give the value of the decay width as

$$\Gamma^{\rm KG}(\pi^0\to\gamma\gamma)=\int\frac{d^3k^{(1)}V}{(2\pi)^3}\frac{d^3k^{(2)}V}{(2\pi)^3}\left(\frac{1}{T}\sum_{\rm spins}\right)|\int d^4x\,\mathfrak{M}^{\rm KG}|^2 \tag{3}$$

$$=|F_\pi|^2m_\pi^3/64\pi. \tag{4}$$

To relate π^0 decay to η decay, one notes that the two-photon state is a pure U-spin-0 state while $\frac{1}{2}|\pi^0\rangle-\frac{1}{2}\sqrt{3}|\eta\rangle$ is U-spin 1.[7] Thus,

$$\langle\gamma\gamma\|\pi^0\rangle=\sqrt{3}\,\langle\gamma\gamma\|\eta\rangle, \tag{5}$$

where the notation $\langle \| \rangle$ in Eq. (5) stands for an appropriate reduced matrix element. Equation (5) is conventionally interpreted to mean that in the SU(3) limit,

$$F_\pi = \sqrt{3}\, F_\eta. \tag{6}$$

Combining Eqs. (6) and (4) with the analog of (4) for $\eta \to \gamma\gamma$ decay, we find

$$R^{\rm KG} \equiv \left[\frac{\Gamma(\eta\to\gamma\gamma)}{\Gamma(\pi^0\to\gamma\gamma)}\right]^{\rm KG} = \frac{1}{3}\left(\frac{m_\eta}{m_\pi}\right)^3 = 22.4, \tag{7}$$

which is in violent disagreement with the experimental value[8,9]

$$R^{\rm expt} = \left[\frac{\Gamma(\eta\to\gamma\gamma)}{\Gamma(\pi^0\to\gamma\gamma)}\right]^{\rm expt} = 129 \pm 33. \tag{8}$$

However, if one uses instead the DKP equation to describe the matrix element, then $[u(p) = (2m_\pi^2)^{-1/2}\,{\rm col}\{ip_0, ip_1, ip_2, ip_3, -m_\pi\}]$

$$\mathcal{L}^{\rm DKP} = (G_\pi/2^{1/2})\psi(x)\epsilon_{\mu\nu\lambda\alpha}\partial_\mu A_\nu^{(1)}\partial_\lambda A_\alpha^{(2)}, \tag{9}$$

$$\psi(x) = (m_\pi/p_0 V)^{1/2} u(p) e^{ip\cdot x}. \tag{10}$$

If one now goes through the rest of the calculation one finds

$$\Gamma^{\rm DKP}(\pi^0\to\gamma\gamma) = |G_\pi|^2 m_\pi^4/64\pi. \tag{11}$$

In obtaining Eq. (11), $u\bar{u}$ in $|\mathfrak{M}^{\rm DKP}|^2$ is evaluated as ${\rm Tr}(u\bar{u})$. (For a detailed discussion of the kinematics of matrix elements involving a single DKP meson see Appendix B of the last Article in Ref. 1.)

The only thing left is to repeat the argument of Eqs. (5) and (6), which now yields

$$G_\pi = \sqrt{3}\, G_\eta, \tag{12}$$

$$R^{\rm DKP} = \left[\frac{\Gamma(\eta\to\gamma\gamma)}{\Gamma(\pi^0\to\gamma\gamma)}\right]^{\rm DKP} = \frac{1}{3}\left(\frac{m_\eta}{m_\pi}\right)^4 = 91.1, \tag{13}$$

which agrees very well with experiment, even before inclusion of the effects of η-η' mixing.

The important point to realize is that simply by using the DKP formulation we have easily found agreement with experiment to within the 10–30% in the matrix element that one expects from SU(3). Contrariwise, the KG pure SU(3) calculation is off by a factor of ~*2.4 in the matrix element.* When we later calculate the small (large) DKP (KG) SU(3) corrections (η-η' mixing) necessary to obtain exact agreement with $R^{\rm expt}$, we will find that they imply very different predictions for $\Gamma(\eta'\to\gamma\gamma)$.

Before proceeding we wish to observe that if the arguments of Eqs. (5), (6), and (12) are literally taken to refer to the physical matrix elements defined from Eqs. (1) and (9), then in the rest frame of the decaying mesons we would have

$$\left[\frac{\mathfrak{M}(\pi^0\to\gamma\gamma)}{\mathfrak{M}(\eta\to\gamma\gamma)}\right]^{\rm KG}_{\vec{p}=0} = \sqrt{3} = \frac{F_\pi}{m_\pi^{1/2}}\frac{m_\eta^{1/2}}{F_\eta}, \tag{14}$$

$$\left[\frac{\mathfrak{M}(\pi^0/\gamma\gamma)}{\mathfrak{M}(\eta\to\gamma\gamma)}\right]^{\rm DKP}_{\vec{p}=0} = \sqrt{3} = \frac{G_\pi}{G_\eta}. \tag{15}$$

So only the DKP Eqs. (12) and (15) are compatible, and not the KG Eqs. (6) and (14).

(2) *Pseudoscalar mixing angle.*—Using the standard sign convention, the mixing of η and η' from the pure octet (η_8) and singlet (η_1) states is given by

$$|\eta\rangle = \cos\theta|\eta_8\rangle - \sin\theta|\eta_1\rangle; \quad |\eta'\rangle = \sin\theta|\eta_8\rangle + \cos\theta|\eta_1\rangle. \tag{16}$$

This means that the predictions for the $P\to\gamma\gamma$ decays are

$$64\pi\Gamma(\pi\to\gamma\gamma) = 3|\mathcal{G}_8|^2(m_\pi)^r, \tag{17}$$

$$64\pi\Gamma(\eta\to\gamma\gamma) = |\mathcal{G}_8|^2\cos^2\theta[1-\tan\theta(\mathcal{G}_1/\mathcal{G}_8)]^2(m_\eta)^r, \tag{18}$$

$$64\pi\Gamma(\eta'\to\gamma\gamma) = |\mathcal{G}_8|^2\cos^2\theta[\tan\theta+\mathcal{G}_1/\mathcal{G}_8]^2(m_{\eta'})^r, \tag{19}$$

$$r = \begin{cases}4\\3\end{cases}; \quad \mathcal{G}_{8,1} = \begin{cases}G_{8,1} & \text{(DKP)}\\ F_{8,1} & \text{(KG)}\end{cases}. \tag{20}$$

To obtain the P mixing angle, one solves

$$\tan 2\theta = [(\eta-\eta')^2 - y^2]^{1/2}/y; \quad y \equiv \eta' + \eta - \tfrac{2}{3}(4K-\pi), \tag{21}$$

where η, η', K, and π are the average isospin multiplet masses [masses squared] for the DKP [KG] solution.[10] Up to a sign, which we shall discuss below,

$$\theta^{\rm DKP} = \mp 23.8°; \quad \theta^{\rm KG} \doteq \mp 10.5°. \tag{22}$$

It is the DKP angle which agrees with the value of approximately $-24°$ obtained from $\pi N \to (\eta, \eta')N$.[11] Also note that the DKP mixing angle is preferable since it is much closer to the "ideal mixing angle," $\theta = \cos^{-1}(\frac{2}{3})^{1/2} = 35°$. That the KG angle is far from the ideal angle has been emphasized by Strocchi and Caffarelli.[12] As the DKP vector mixing angle is 36.8°, both of the DKP angles are near the ideal angle.

(3) *Sign of the mixing angle and $\eta'\to\gamma\gamma$ decay.*—To obtain a prediction for $\Gamma(\eta'\to\gamma\gamma)$ we first divide Eqs. (18) and (19) by Eq. (17). Then using $\theta^{\rm DKP}$ [$\theta^{\rm KG}$] from Eq. (22) we solve for the value of G_1/G_8 [F_1/F_8] needed to obtain agreement with the

experimental value of $\Gamma(\eta\to\gamma\gamma)/\Gamma(\pi^0\to\gamma\gamma)$. Since the theoretical expression for $\Gamma(\eta'\to\gamma\gamma)/\Gamma(\pi^0\to\gamma\gamma)$ depends only on these (now determined) values of G_1/G_8 $[F_1/F_8]$ and θ^{DKP} $[\theta^{\mathrm{KG}}]$, $\Gamma(\eta'\to\gamma\gamma)$ can now be predicted directly from a knowledge of $\Gamma^{\mathrm{expt}}(\pi^0\to\gamma\gamma)$ and $\Gamma^{\mathrm{expt}}(\eta\to\gamma\gamma)$. In solving for $\mathcal{G}_1/\mathcal{G}_8$ we obtain two solutions, first by using one sign of $\mathcal{G}_1/\mathcal{G}_8$ relative to $\tan\theta$, and then the other. These and all our other results are shown in Table I. The errors come mainly from uncertainties in $\Gamma_{\mathrm{tot}}^{\mathrm{expt}}(\pi^0)$ and $\Gamma_{\mathrm{tot}}^{\mathrm{expt}}(\eta)$.

We can immediately rule out the solutions with $(\mathcal{G}_1/\mathcal{G}_8)\tan\theta>0$ for both DKP and KG. This is because new experiments by Basile *et al.*[13] and Harvey *et al.*[14] have precisely determined the branching ratio of $\eta'\to\gamma\gamma$ to be[8] 0.0184 ± 0.0035 which means that[8] $\Gamma^{\mathrm{expt}}(\eta'\to\gamma\gamma)<74\pm14$ keV. Further, a new experiment[15] gives $\Gamma^{\mathrm{expt}}(\eta'\to\gamma\gamma)<15_{-15}^{+19}$ keV. These numbers clearly rule out the solutions in Table I with $(\mathcal{G}_1/\mathcal{G}_8)\tan\theta>0$, and also those theoretical KG models which have predicted $\Gamma(\eta'\to\gamma\gamma)$ is O(100's of keV).

The allowed $[(\mathcal{G}_1/\mathcal{G}_8)\tan\theta<0]$ DKP and KG predictions for $\Gamma(\eta'\to\gamma\gamma)$ are very distinct. For KG there is a very large value of F_1/F_8 and $\Gamma^{\mathrm{KG}}(\eta'\to\gamma\gamma)=52\pm24$ keV. This last figure is at about the upper experimental limit just quoted, and could soon be ruled out by a better experiment. Contrariwise the DKP solution is $\Gamma^{\mathrm{DKP}}(\eta'\to\gamma\gamma)=0.32_{-0.32}^{+0.99}$ keV; i.e., O(100's of eV). The large uncertainty arises from the sensitivity of Eq. (19) to the uncertainty in determining G_1/G_8. Even so, however, the DKP prediction is clearly at least 1–2 orders of magnitude lower than the KG prediction, and so offers a clear test between the two formulations.

We observe that since the branching ratio of $\eta'\to\gamma\gamma$ is now fairly well known, the DKP prediction for $\Gamma(\eta'\to\gamma\gamma)$ can be used to compute $\Gamma_{\mathrm{tot}}(\eta')$ and $\Gamma(\eta'\to\eta\pi\pi)$. We find

$$\Gamma_{\mathrm{tot}}^{\mathrm{DKP}}(\eta')=17_{-17}^{+54}\ \mathrm{keV};\qquad \Gamma^{\mathrm{DKP}}(\eta'\to\eta\pi\pi)=12_{-12}^{+37}\ \mathrm{keV}. \tag{23}$$

The width for the *strong* decay $\eta'\to\eta\pi\pi$ in Eq. (23) is comparable to that for the second-order electromagnetic decay $\eta\to3\pi$, $\Gamma^{\mathrm{expt}}(\eta\to3\pi)=1.42\pm0.31$ keV, which has very nearly the same Q value (144 MeV for $\eta\to3\pi^0$ versus 138 MeV for $\eta'\to\eta\pi^0\pi^0$). This indicates that the $\eta'\to\eta\pi\pi$ decay amplitude is strongly suppressed (by a factor of ~30) compared to what one would expect for a typical strong decay.

The sensitivity of the DKP solution to G_1/G_8 arises because $G_1/G_8=O(1)$, which is far from true for the much larger KG value of F_1/F_8. The lower (DKP) value of $\mathcal{G}_1/\mathcal{G}_8$ is not at all surprising. If we calculate $\mathcal{G}_1/\mathcal{G}_8$ from the pole model $(\eta_1$ or $\eta_8)\to(2\rho$ and $2\omega)\to\gamma\gamma$ we find

$$\mathcal{G}_1/\mathcal{G}_8\cong+\tfrac{1}{3}\sqrt{2}=+0.47, \tag{24}$$

where we have used the $\tilde{\mathrm{U}}(12)$ model of Sakita and Wali[16] to estimate the $\rho\rho\eta_{1,8}$ and $\omega\omega\eta_{1,8}$ coupling constants in the limit of no η_1-η_8 mixing. Note that Eq. (24) gives the desired sign $\mathcal{G}_1/\mathcal{G}_8>0$ [implying $\theta<0$ since experiment gives $(\mathcal{G}_1/\mathcal{G}_8)\tan\theta<0$] and yields $R^{\mathrm{DKP}}(\tilde{\mathrm{U}}(12))=111$, in excellent agreement with experiment.

We emphasize that the uncertainties that remain in determining $\mathcal{G}_1/\mathcal{G}_8$ and $\Gamma(\eta'\to\gamma\gamma)$ are almost entirely due to the experimental errors[8] in three quantities: (1) 12% error in $\Gamma_{\mathrm{tot}}^{\mathrm{expt}}(\pi^0)$; (2) 22% error in $\Gamma_{\mathrm{tot}}^{\mathrm{expt}}(\eta)$; (3) the limit $\Gamma_{\mathrm{tot}}^{\mathrm{expt}}(\eta')<$[4 MeV,[8] or 1.9 MeV[18]]. We strongly encourage experiments to settle these questions.

In conclusion we point out that the decays ρ

TABLE I. Comparison of the KG and DKP predictions with experiment for $\Gamma(\pi^0,\eta,\eta'\to\gamma\gamma)$ and the pseudoscalar mixing angle. For the mixing angle θ, the DKP [KG] prediction is obtained from the mass [mass-squared] formula.

	Pure SU(3) (no η-η' mixing) predictions for $\frac{\Gamma(\eta\to\gamma\gamma)}{\Gamma(\pi^0\to\gamma\gamma)}$	Pseudoscalar η-η' mixing angle, θ	Predictions for $\Gamma(\eta'\to\gamma\gamma)$ with η-η' mixing: $(\mathcal{G}_1/\mathcal{G}_8)\tan\theta>0$ solution, experimentally ruled out		$(\mathcal{G}_1/\mathcal{G}_8)\tan\theta<0$ solution, experimentally allowed	
			$\lvert\mathcal{G}_1/\mathcal{G}_8\rvert$	$\Gamma(\eta'\to\gamma\gamma)$ keV	$\lvert\mathcal{G}_1/\mathcal{G}_8\rvert$	$\Gamma(\eta'\to\gamma\gamma)$ keV
Klein-Gordon	22.4	$\mp10.5°$	18.64 ± 1.68	316 ± 68	7.80 ± 1.68	52 ± 24
Duffin-Kemmer-Petiau	91.1	$\mp23.8°$	5.22 ± 0.37	176 ± 32	0.68 ± 0.37	$0.32_{-0.32}^{+0.99}$
Experiment	129 ± 33 [a]	See Ref. 11		$\Gamma<74\pm14$ [a] $\Gamma<15_{-15}^{+19}$ [b]		$\Gamma<74\pm14$ [a] $\Gamma<15_{-15}^{+19}$ [b]

[a] Ref. 8.
[b] Ref. 15.

$\rightarrow \pi\gamma$, $\omega \rightarrow \pi\gamma$, and $K^* \rightarrow K\gamma$ can be treated in a similar fashion, and will be discussed elsewhere. Although the branching ratio for $\omega \rightarrow \pi\gamma$ is known, those for $\rho \rightarrow \pi\gamma$ and $K^* \rightarrow K\gamma$ are not, since the photonic decay modes of ρ and K^* are difficult to observe directly. However, these modes can be determined indirectly by measuring the differential cross sections for the production of the respective vector mesons by a π or K incident on a high-Z nucleus.[20]

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[10]In principle we could obtain four fits by using both the mass and mass-squared mixing angles in both the DKP and KG formulations. However, with the usual type of Lagrangians, it is reasonable to associate only the mass [mass-squared] formula with the DKP [KG] formulation.

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