Gravitational Synchrotron Radiation in the Schwarzschild Geometry*

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The existence of a mechanism for gravitational synchrotron radiation is demonstrated in solutions of the wave equation in the Schwarzschild background, with the source a particle in a highly relativistic circular geodesic. The main features (high-frequency harmonics, narrow angular distribution in latitude) are shown to hold for vector (electromagnetic) and tensor (gravitational) radiations, which are expected to be strongly polarized in the orbit plane. Detailed formulas for the spectrum are given in the scalar case.

As described in the preceding Letter,[1] gravitational synchrotron radiation (GSR) is a crucial concept in searching for exotic astrophysical phenomena, which it might make visible through Weber's[2] gravity telescopes. *We here demonstrate that a gravitational synchrotron radiation mechanism exists as a consequence of Einstein's general relativity theory.* Thus, particles in highly relativistic circular orbits in gravitational fields can be expected to radiate strongly, into limited angles, at high harmonics of the orbit frequency. Charged particles would radiate electromagnetic waves predominantly, while a neutral particle would radiate gravitational waves. One example, however, suffices to prove the existence of the GSR phenomenon; so this first computation assumes, for simplicity, that the particle couples to a massless scalar field and radiates scalar waves.

In this paper we consider a highly relativistic (unstable) circular geodesic orbit near $r = 3M$ in the Schwarzschild geometry (in units with $G = c = 1$). Although this calculation is astrophysically unrealistic, it is important to demonstrate the possibility of GSR in principle. Furthermore, the situations of possible astrophysical interest,[1] which involve the extreme ($a \simeq M$) Kerr[3] black hole metrics, are expected to have features in common with Schwarzschild GSR. Gravitational radiation from highly relativistic orbits has been computed previously by Peters,[4] who also found radiation concentrated in narrow angles. Peters's relativistic gravitational bremsstrahlung calculations considered scattering at large impact parameters $b \gg M$ and for small scattering angles $\psi \ll 1$, while the present GSR calculations in the Schwarzschild metric may be considered to treat high-energy scattering at small impact parameters $b \simeq 3\sqrt{3}M$ which involve very large scattering angles $\psi \gg 2\pi$. (The scattering orbit winds around many times near its pericenter, and is approximated by an unstable circular orbit.)

The qualitatively most significant aspect of our result is a barrier penetration factor arising in the solution of a homogeneous, Schrödinger-like, wave equation. But the homogeneous equations are identical in the scalar, vector, and tensor cases to the required accuracy. Thus the features controlled by this factor will hold also for radiation of electromagnetic (vector) and gravitational (tensor) waves. These features are (i) a spectrum cutoff for frequencies $\omega > \omega_{\text{crit}}$, and (ii) a suppression of radiation in angular (Y_l^m) modes with $q \gg 1$ where

$$2q \equiv l - |m| . \tag{1}$$

The barrier penetration factor which gives these results is just

$$\exp[-\tfrac{1}{2}\pi(1 + |m|\delta) - 2\pi q] , \tag{2}$$

when δ is small [see Eq. (19) which gives the power output in the lm mode]. Here δ is related to the radius r_0 of a circular geodesic orbit in the field of a Schwarzschild mass M by

$$r_0 = (3 + \delta)M , \tag{3a}$$

and to the energy (as measured at infinity) per unit rest mass γ by

$$\delta^{-1} \simeq 3\gamma^2 . \tag{3b}$$

The radiation in the Y_l^m mode is found to be emitted at frequency

$$\omega = m\omega_0 , \tag{4}$$

where

$$\omega_0 = (M/r_0^3)^{1/2} \tag{5}$$

is the fundamental (orbit) frequency. Thus the barrier factor (2) lets us define

$$\frac{\omega_{\text{crit}}}{\omega_0} = m_{\text{crit}} = \frac{4M}{\pi(r_0 - 3M)} = \frac{4}{\pi\delta} = \frac{12}{\pi}\gamma^2 , \tag{6}$$

which shows $m_{\rm crit} \gg 1$ as $r_0 \to 3M$. Since Y_l^m contains a factor $\sin^{|m|}\theta = \cos^{|m|}\vartheta \simeq \exp(-\frac{1}{2}|m|\vartheta^2)$, all modes which radiate a given, high, harmonic $m \gg 1$ have their radiation intensity $(\propto |Y_l^m|^2)$ concentrated in a beam of half-width

$$\Delta\vartheta = |m|^{-1/2} \tag{7}$$

centered on the equatorial plane $\vartheta \equiv \frac{1}{2}\pi - \theta = 0$. Although the detailed computations necessary to study polarizations have not been completed, it is expected that electromagnetic and gravitational waves emitted by particles in highly relativistic circular geodesic orbits will be strongly polarized in the plane of the orbit.

In special relativity the interaction between a scalar field φ and a point particle of mass μ and world line $z^\alpha(\lambda)$ is defined by the action integral (signature $-+++$)

$$I = -(8\pi)^{-1}\int \eta^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta}d^4x - \mu\int(1+f\varphi)(-\eta_{\alpha\beta}\dot{z}^\alpha\dot{z}^\beta)^{1/2}d\lambda . \tag{8}$$

The above action leads to static scalar forces equal to the Newtonian gravitational force between pairs of particles if each has "scalar charge" $f = \sqrt{G}$.

This value of f is useful in extrpolating scalar GSR to the gravitational (tensor) case with one proviso: The source in the scalar wave equation is

$$(-gT)^{1/2} = \mu\int d\tau\, u^\nu u_\nu \delta^4(x^\alpha - z^\alpha(\tau)) = -\mu(dz^0/d\tau)^{-1}\delta^3(x - z(t)),$$

while in the tensor case it is

$$(-gT^{\alpha\beta})^{1/2} = \mu(u^\alpha u^\beta/u^0)\delta^3(x - z(t)),$$

so gravitational wave amplitudes should be stronger by a factor of about $(u^0)^2 = (dz^0/d\tau)^2$ than the corresponding $(f=\sqrt{G})$ scalar amplitudes when $|\beta^k| = |u^k/u^0| \sim 1$, giving a ratio of approximately $(u^0)^4$ in intensities. In order to use the present calculations to estimate the intensity of scalar radiation which would be emitted in the Brans-Dicke[5] theory, where a fraction $(4+2\omega)^{-1}$ of the Newtonian force is due to the scalar field, one could set $f = G^{1/2}(4+2\omega)^{1/2}$, where $\omega \geq 6$ is the Brans-Dicke coupling constant.

The equation we propose to solve then is the wave equation following from the general relativistic form of Eq. (8),

$$(\partial/\partial x^\alpha)[(-g)^{1/2}g^{\alpha\beta}(\partial\varphi/\partial x^\beta)] = -4\pi f(-g)^{1/2}T = 4\pi f\mu(u^0)^{-1}\delta^3(\vec{x} - \vec{z}(t)), \tag{9}$$

with the $g_{\alpha\beta}$ chosen to be the Schwarzschild metric for a mass M, and the orbit $z(t)$ of the source given by $\theta = \frac{1}{2}\pi$, $r = r_0 = \text{const}$, and $\varphi = \omega_0 t$. For a geodesic orbit one has

$$u^0 \equiv dt/d\tau = (1 - 3Mr_0^{-1})^{-1/2} \simeq (3/\delta)^{1/2} \simeq 3\gamma \tag{10}$$

as well as in Eq. (5). The expansion

$$\varphi = \sum_{m=-\infty}^{+\infty}\sum_{l=|m|}^{\infty}(r)^{-1}u_{lm}(r)Y_l^m(\theta,\varphi)\exp(-im\omega_0 t), \tag{11}$$

in which the Y_l^m are spherical harmonics, leads to a radial equation

$$-\frac{d^2u}{dr^{*2}} + \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right)u - m^2\omega_0^2 u = C\delta(r^* - r_0^*), \tag{12}$$

which employs the Regge-Wheeler[6] coordinate $r^* = r - 3M + 2M\ln(rM^{-1} - 2)$ and in which $u \equiv u_{lm}$ and

$$C \equiv C_{lm} = -4\pi f\mu(r_0u^0)^{-1}Y_l^m(\tfrac{1}{2}\pi, 0). \tag{13}$$

This equation is well studied in connection with the Schwarzschild metric.[7,8] Similar equations would be obtained in the electromagnetic[9] and gravitational wave[6,10] cases. These equations all have the form

$$-u'' + (V - E)u = C\delta(r^* - r_0^*), \tag{14}$$

with $E = m^2\omega_0^2$, but differ in the values of C and V. The forms for V are known from the studies cited of the homogeneous equations ($C=0$), and all have the common form

$$V = (1 - 2Mr^{-1})l(l+1)r^{-2} + O(l^0) \tag{15}$$

for large l. [The even and odd tensor potentials differ from each other only in order l^{-2}; for the vector case $O(l^0) \equiv 0$.]

The solution of Eq. (12) is obtained by matching solutions of the homogeneous equation at $r^* = r_0^*$ and imposing the Matzner[7] boundary conditions: purely outgoing radiation for $r^* \to +\infty$, purely ingoing (down the hole) radiation for $r^* \to -\infty$. The total power output in mode l, m can then be put in the form

$$P_{\text{out}}(l,m) = (1/8\pi)|C|^2|L(r_0^*)|^2, \tag{16}$$

where $L(r^*)$ is a scattering solution of the homogeneous equation for a wave of the form $1 \times \exp(-im\omega_0 r^*)$ incident from $r^* = \infty$. In the limit $l \gg 1$ in which we are interested, the wave function $L(r^*)$ can be obtained by WKB methods as sketched in Fig. 1. For small δ and for $|m| \gg q$, the main spectrum requires $L(r^*)$ near the maximum of the potential barrier in Eq. (14). The methods of Ford, Hill, Wakano, and Wheeler[11] can be used and give the wave function in terms of parabolic cylinder functions $L(r^*) \propto D_{-1/2-i\epsilon/2} \times (-\eta r^*)$. Here

$$\epsilon = 1 + 4q + |m|\delta = 1 + 4q + (4/\pi)\omega/\omega_{\text{crit}} \tag{17}$$

is the barrier integral $(4/\pi)\int (V-E)^{1/2}dr^*$ from the orbit r_0^* to the outer turning point r_+^*, and $\eta = (1-i)[l(l+1)]^{1/4}(3M)^{-1}$. For small δ (i.e., large γ), the limit $r_0^* \simeq 3M\delta = M/\gamma^2 \to 0$ allows the parabolic cylinder functions to be evaluated in terms of Γ functions. The resultant formula for the radiated power is $P_{\text{out}} = 0$ when $2q = l - |m|$ is odd, and reads

$$P_{\text{out}} = f^2\left(\frac{\mu}{M}\right)^2 \frac{32}{27}(4\pi)^{-5/2}\frac{(2q)!}{2^{2q}(q!)^2}\left|\frac{m}{m_{\text{crit}}}\right| \times e^{-\pi\epsilon/4}|\Gamma(\tfrac{1}{4}+\tfrac{1}{4}i\epsilon)|^2 \tag{18}$$

for $q = 0, 1, 2, \cdots$ and $|m| \gg q$. To find the total power $P_{\text{out}}(\omega)$ radiated at a given frequency $\omega = |m|\omega_0$, Eq. (18) must be summed over q; the

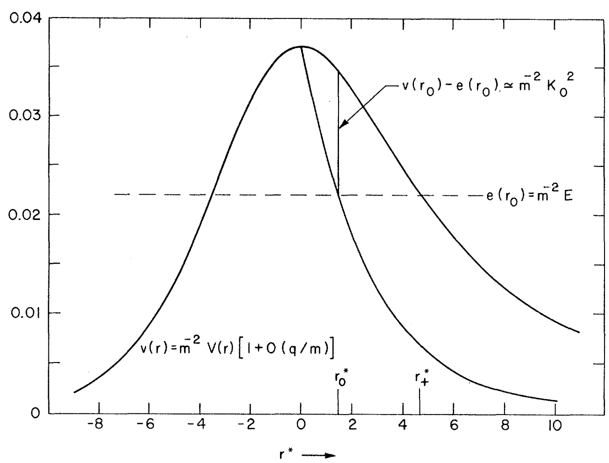

FIG. 1. Reduced ($m \to \infty$) potential and energy. For high l and $m = |m|$, the energy E and potential V of Eqs. (12) and (14) are conveniently written $V(r) = m^2 v(r) \times [1 + m^{-1}(4q+1) + m^{-2}(2q)(2q+1)]$ and $E = m^2 e(r_0)$, where $v(r) = r^{-3}(r-2)$, $e(r) = r^{-3}$, and $l = m + 2q$. (Units $G = 1 = c$, and $M = 1$). For any chosen orbit radius r_0, the turning point r_+, defined by $V(r_+) = E$, can be read from this plot to within $O(q/m)$ errors as the solution of $v(r_+) = e(r_0)$. $L(r^*)$ is then computed at r_0^* as $L(r_0^*) = E^{1/4} \times \kappa_0^{-1/2} e^{-\Theta} e^{-i\pi/4}$, where $\kappa_0 \equiv [V(r_0) - E]^{1/2}$ and

$$\Theta \equiv \int_{r_0^*}^{r_+^*} [V(r) - E]^{1/2} dr^*.$$

Validity of this standard WKB solution requires $\Theta \gg 1$. For a rough estimate, use $\Theta \sim \kappa_0(r_+^* - r_0^*) \simeq m[V(r_0) - e(r_0)]^{1/2}(r_+^* - r_0^*)$. This is clearly large for large m except at $r_0^* = 0$, where $v \to e$ and $r_+^* \to r_0^*$. Thus, there is no GSR (strong high m emission) except for $r_0^* \to 0$ which is $r_0 \to 3$. As $r_0^* \to 0$, the $O(q/m)$ term in $m^{-2}V$ must be retained, and $\kappa_0 = m(3)^{-3/2}[\delta + m^{-1}(4q+1)]^{1/2}$, where $\delta = r_0 - 3$, and $r_+^* - r_0^* \simeq (3)^{3/2}[\delta + m^{-1}(4q+1)]^{1/2}$ so $\Theta \sim \kappa_0(r_+^* - r_0^*) = \epsilon \equiv 1 + 4q + m\delta$. Thus there is a dominant barrier penetration factor $e^{-\Theta}$ only for $q \gg 1$ or for $m \gg \delta^{-1} = 3\gamma^2$.

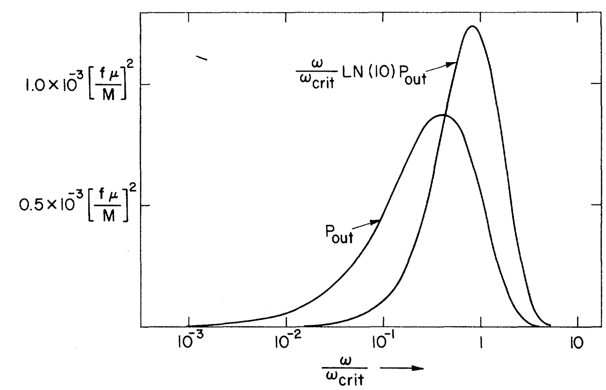

FIG. 2. Power radiated out to infinity in GSR in the limit $\delta = (3\gamma^2)^{-1} \ll 1$. Here

$$P_{\text{out}}(m) = \sum_{l=m}^{\infty} P_{\text{out}}(l,m)$$

for scalar radiation is given as a function of $\omega/\omega_{\text{crit}} = \frac{1}{4}\pi m\delta$ for each frequency harmonic $m = \omega/\omega_0 \gg 1$. The total power emitted in all harmonics is

$$P_{\text{tot}} = \sum_{m=0}^{\infty} P_{\text{out}}(m) \simeq \int_0^\infty P_{\text{out}}\, dm.$$

Since this can be written $P_{\text{tot}} = \int (2\pi/\omega_0)P_{\text{out}}\, d\nu$, the power emitted in a unit frequency interval is $dP_{\text{out}}/d\nu = (2\pi/\omega_0)P_{\text{out}}$. To obtain P_{tot} in terms of the area under a curve in a semilog plot where the abscissa is really $\log_{10}(\omega/\omega_{\text{crit}})$, one writes

$$P_{\text{tot}} = \frac{\omega_{\text{crit}}}{\omega_0}\int_0^\infty \frac{\omega}{\omega_{\text{crit}}}(\ln 10)P_{\text{out}}\, d\left(\log_{10}\frac{\omega}{\omega_{\text{crit}}}\right).$$

The integrand is plotted to show that the bulk (82%) of the energy is emitted in the decade $10^{-0.7} < \omega/\omega_{\text{crit}} < 10^{+0.3}$. Numerical integration gives $P_{\text{tot}} = f^2(\mu\gamma/M)^2(3.9 \times 10^{-3})$. Here P_{out} and P_{tot} are dimensionless and should be multiplied by $1 = c^5/G = 3.63 \times 10^{59}$ erg/sec $= 2.03 \times 10^5 M_\odot c^2$/sec to obtain their values in other units.

result is shown in Fig. 2, where the $q=0$ term contributes over 99.9% at all frequencies. If the barrier penetration integral is large, $\epsilon \gg 1$, then one obtains an asymptotic form

$$P_{\text{out}}(l, m) = f^2 \left(\frac{\mu}{M}\right)^2 \frac{32}{27} (4\pi)^{-3/2} \frac{(2q)!}{2^{2q}(q!)^2} \times \left|\frac{m}{m_{\text{crit}}}\right| \frac{e^{-\pi\epsilon/2}}{\epsilon^{1/2}}. \qquad (19)$$

This form can also be obtained directly, using standard[12] WKB methods. It shows both the high-frequency cutoff since $\epsilon \simeq |m|\delta \to \infty$ as $\omega/\omega_0 = |m| \to \infty$, and the $q \gg 1$ cutoff (angular beaming) since $\epsilon \simeq 4q \to \infty$ for $q \to \infty$. The amount of radiation which goes down the black hole is given by $P_{\text{down}}(l, m) = (8\pi)^{-1}|C|^2|R(r_0^*)|^2$, where $R(r^*)$ is a scattering wave function with an incident wave moving to the right, $R(r^*) \propto D_{-1/2-i\epsilon/2}(+\eta r^*)$. In the $\delta \to 0$ limit used in Eqs. (16)–(19), where $r_0^* \to 0$, one then finds $P_{\text{down}} = P_{\text{out}}$.

The azimuthal distribution of GSR could be studied by summing the series $\sum u_{ll} Y_l^{\,l}(\theta, \varphi) \times \exp(-il\omega_0 t)$ in some approximation. Calculations based on geodesic arguments[13] show that an analog of the "rotating searchlight effect" occurs and gives a narrow peaking of the field as a function of $\varphi - \omega_0 t$ as in ordinary synchrotron radiation.

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[1]C. W. Misner, preceding Letter, Phys. Rev. Lett. 28, 994 (1972).

[2]J. Weber, Phys. Rev. Lett. 22, 1320 (1969), and 24, 276 (1970), and 25, 180 (1970), and Sci. Amer. 224, 22 (1971).

[3]R. P. Kerr, Phys. Rev. Lett. 11, 237 (1963).

[4]P. C. Peters, Phys. Rev. D 1, 1559 (1970).

[5]C. Brans and R. H. Dicke, Phys. Rev. 124, 925 (1961).

[6]T. Regge and J. A. Wheeler, Phys. Rev. 108, 1063 (1957).

[7]R. A. Matzner, J. Math. Phys. (N.Y.) 9, 163 (1968).

[8]R. H. Price, Phys. Rev. D (to be published).

[9]J. A. Wheeler, Phys. Rev. 97, 511 (1955).

[10]F. J. Zerilli, Phys. Rev. Lett. 24, 737 (1970).

[11]D. W. Ford *et al.*, Ann. Phys. (New York) 7, 239 (1959).

[12]See, for example, E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1970), 2nd ed., Chap. 7.

[13]H. G. Hughes, III, to be published.

Photon Mass and New Experimental Results on Longitudinal Displacements of Laser Beams near Total Reflection

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Recent observations by Imbert to test predictions and observations of Goos and Hänchen by reflecting a laser beam near the total reflection angle contradicts classical and quantum predictions for the longitudinal displacements of linearly polarized incident plane waves. They can be simply interpreted as a Stern-Gehrlach type of measurement of the photon spin, provided the photon has a nonzero rest mass.

Recent observations by Mazet, Imbert, and Huard[1] on total reflection, which appear to contradict classical expectations, have led the authors[2] to a possible interpretation in terms of a nonzero photon mass. In this Letter we want to discuss this contradiction in terms of the quantum theory of radiation and generalize our proposal to the quantum theory of massive photons considered as true spin-1 particles.

To our knowledge the best quantum analysis of the longitudinal displacement of an incident light beam near total reflection (the Goos-Hänchen effect[3]) has been proposed by Agudin[4] on the basis of time delay of scattering processes based on the theories of Wigner,[5] Smith,[6] and Froissart, Goldberger, and Watson.[7]