an experiment similar to the one described above (5 mfp thick) on the first of the High Energy Astronomical Observatories satellites. This experiment will measure the spectra of all cosmic rays up to 10^{14} eV. The spectrometer to be flown will be deeper, and the improved statistics from the planned two-year exposure will help to resolve the discrepancy at 1000 GeV.

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Generalization of the Concept of Invariance of Differential Equations. Results of Applications to Some Schrödinger Equations

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We have found that differential equations can be form invariant under a larger class of infinitesimal transformations than those considered by Lie and Ovsjannikov. We give a generalization of the concept of point transformation. It is necessary for the systematic determination of the generators of continuous invariance groups of, e.g., the partial differential equations of physics. Applying it to Schrödinger's equation, time-dependent constants of the motion are found systematically, as illustrated here for the hydrogen atom.

Lie[1] based his group theoretical treatment of systems S of differential equations (linear and/or nonlinear) of order π,

$$F^r(x,u,\partial_j u,\partial_j\partial_k u,\cdots)=0;\quad r=1,2,\cdots,R, \tag{1}$$

upon the concept of an infinitesimal point transformation of an N-dimensional Euclidean manifold

$$(x,u)\to(\bar{x},\bar{u}) \tag{2}$$

with coordinates of two types,[1]

$$x=(x^1,x^2,\cdots,x^n) \tag{3a}$$

and

$$u=[u^1(x),u^2(x),\cdots,u^m(x)]. \tag{3b}$$

The functions u^j may be taken to be the original unknown functions appearing in S, or they may be new functions that have arisen by reduction of S to an "equivalent" set S' of quasilinear first-order equations through the device of introducing unknown functions for the various derivatives appearing in S.[2]

Lie's transformations are of the form

$$\bar{x}^i = f^i(x,u;\ \delta a),\ \text{with } x^i = f^i(x,u;0);$$
$$\bar{u}^j(\bar{x}) = g^j(x,u;\ \delta a),\ \text{with } u^j(x) = g^j(x,u;0). \tag{4}$$

We have found that when dealing with partial differential equations it is necessary to enlarge the concept of invariance usually adopted[2] (cf. the discussion below). We generalize the infinitesimal point transformations of Lie to infinitesimal transformations of the space obtained by augmenting the original coordinates with their derivatives with respect to the x^i; that is, we let

$$\bar{x}^i = f^i(x,u,\partial_j u,\partial_j\partial_k u,\cdots;\ \delta a),$$
$$\bar{u}^i(\bar{x}) = g^i(x,u,\partial_j u,\partial_j\partial_k u,\cdots;\delta a). \tag{5}$$

The total changes in the x^i and u^j due to variation of the parameter a are thus

$$\delta x^i = (df^i/da)_{a=0}\,\delta a \equiv \xi^i(x,u,\partial_j u,\partial_j\partial_k u,\cdots)\,\delta a,\quad \delta u^i = (dg^i/da)_{a=0}\,\delta a \equiv \eta^i(x,u,\partial_j u,\partial_j\partial_k u,\cdots)\,\delta a. \tag{6}$$

We define $\eta_j{}^i$ by

$$\partial\bar{u}^i(\bar{x})/\partial\bar{x}^j = \partial u^i/\partial x^j + \delta a\,\eta_j{}^i, \tag{7a}$$

and $\eta_{jk}{}^i$ by

$$\partial^2\bar{u}^i(\bar{x})/\partial\bar{x}^j\partial\bar{x}^k = \partial^2 u^i/\partial x^j\partial x^k + \delta a\eta_{jk}{}^i\ ,\ \text{etc.} \tag{7b}$$

If Eqs. (7) are form invariant under the generalized transformations (5), then YF^r, $r=1,2,\cdots,R$, must vanish, where

$$Y = \xi^i\frac{\partial}{\partial x^i} + \eta^j\frac{\partial}{\partial u^j} + \eta_k{}^j\frac{\partial}{\partial(\partial_k u^j)} + \eta_{jk}{}^i\frac{\partial}{\partial(\partial_j\partial_k u^i)} + \cdots\ . \tag{8}$$

Osvjannikov has extended Lie's treatment to partial differential equations of arbitrary order in the following manner[2]. He defines the Lie operators

$$X = \xi^i(x,u)\,\partial/\partial x^i + \eta^j(x,u)\,\partial/\partial u^j \tag{9}$$

and its extensions $\tilde{X}, \tilde{\tilde{X}}$, etc. He then considers a system S of differential equations in which the highest-order derivative that appears is of order π. He proposes that invariance of the system S with respect to X is to be understood in the sense of invariance with respect to the operator obtained by extension of X up to the order π. He then develops two alternative methods of finding such invariants. One of these, set forth in detail for systems of linear second-order partial differential equations, involves a direct action of X and its extensions upon the equations of interest. In the other approach, to avoid consideration of high extensions of X, he replaces the system S by an "equivalent" quasilinear S' having partial derivatives of the first order only, by introducing new unknown functions and, if required, differentiations.

In this section we wish to show that for partial differential equations our transformations subsume Osvjannikov's, while for ordinary differential equations and their derived systems of first-order linear equations they are equivalent to Lie's transformation, as long as the highest derivative in the ordinary differential equation can be uniquely isolated. Then

$$\frac{d^\pi u^1}{dx^\pi} = W\left(x, u^1, \frac{du^1}{dx}, \frac{d^2u^1}{dx^2}, \cdots, \frac{d^{\pi-1}u^1}{dx^{\pi-1}}\right), \tag{10}$$

and derivatives higher than the πth can be expressed in terms of the πth and lower derivatives. Derivatives higher than πth are then redundant in (1). When one converts an ordinary differential equation to a set of first-order partial differential equations by substituting new variables for the derivatives, the number of independent functions u^i that can be introduced is for the same reason strictly limited, and a strict equivalence can be established between the ordinary differential equation and the set of derived first-order partial differential equations. The two approaches are equivalent and our transformations give the same results as those of Lie.

Suppose, however, that the differential equation(s) of interest are of the general form (1). In this case one may again lay down a set of quasilinear defining equations, but though the original system may provide additional relations among the derivatives, these relations are not sufficient in general to make derivatives of arbitrarily high order redundant. There is, therefore, no *a priori* reason for supposing that arbitrary functions of these derivatives cannot occur in an invariant transformation of the original system S. Furthermore, it is seen that there is no longer a unique correspondence between the original differential equations and a quasilinear set of first-order equations. The concept of a set S' of a finite number of such equations "equivalent" to the original equation or set S of equations is, therefore, at best ambivalent, and transformations of the form (5) do not in general reduce to Lie's form (4).

As a consequence, our more general concept of the invariance of differential equations to infinitesimal transformation removes three essential limitations of Osvjannikov's formulation:

(i) In his first formulation there is no direct way to obtain invariants that convert the u^k to their second- and higher-order derivatives with respect to the x^i, or to functions of these derivatives. Such invariants are, however, of widespread occurrence.

(ii) In order to obtain such invariants using Osvjannikov's method, one follows his lead and replaces the given higher-order partial differential equation(s) by a set of quasilinear first-order equations, then uses his second method. However, if one stops at the point where derivatives of order no higher than π appear, then one may miss important invariants of the equations.

(iii) Ovsjannikov proves that the commutation relations of a set of invariant operators, $X_1, X_2, \cdots$ are the same as those of the extended operators.[2] In contrast, in our approach, if one considers a heirarchy of ever more general Y operators, $Y_i, Y_i, Y_i, \cdots$, which are allowed to depend upon successively higher derivatives, the local Lie groups obtained are not, in general, isomorphic.

Let K be a linear differential operator such that

$$K\psi(x)=0, \quad x=(x^1, x^2, \cdots, x^n). \tag{11}$$

Then, for a wide class of operators in the space of solutions ψ,

$$Q=q(x)+q^k(x)\partial_k+q^{kj}(x)\partial_k\partial_j+\cdots. \tag{12}$$

If Q is an infinitesimal invariant, then

$$KQ\psi=0. \tag{13}$$

This is a differential equation determining Q, and it can be solved by the classical method of Lie.[1] It yields solutions of the form $Q=\gamma^m Q_m$, where the γ^m are arbitrary constants and the Q_m are of the form (12).

We treat time-dependent Schrödinger equations as an example ($K=H-i\partial_t$, where H is the Hamiltonian) and outline one method for determining invariants involving an infinite number of derivatives—a method which is useful when the spectrum of H is known. In this, the original equation is subjected to a similarity transformation with operator $D=D_rD_t$. The general theory,[4] a quantum analog of Hamilton-Jacobi theory, specifies these operators, which are chosen to transform to an equation which has no negative-order derivatives and whose spectrum is linear in the quantum numbers determining the original energy. For the hydrogen atom, D takes the form

$$D=D_rD_t=\exp\left[r\partial_r\ln\left(\frac{-i}{2Z}\partial_t\right)\right]\exp\left\{t\partial_t\ln\left[\frac{2Z}{(-2H)^{3/2}}\right]\right\}. \tag{14}$$

A straightforward application of the methods of Lie[1] to the transformed equation yields a fifteen-parameter Lie group. Transforming back to the generators of the original equation, we obtain, in addition to the angular momentum $\vec{L}$ and the Runge-Lenz vector $\vec{A}$ with components

$$Q_1=L_x,\quad Q_2=L_y,\quad Q_3=L_z,\quad Q_4=A_x,\quad Q_5=A_y,\quad Q_6=A_z, \tag{15a}$$

the following time-dependent generators:

$$Q_{7,8}=(Q_9)^{-2}D^{-1}e^{\pm it}(\pm ir\partial_r+\partial_t-\tfrac{1}{2}ir\pm i)D(Q_9)^{+2},\quad Q_9=D^{-1}\partial_tD,\quad Q_{10}=[Q_4,Q_7],\quad Q_{11}=[Q_5,Q_7],$$
$$Q_{12}=[Q_6,Q_7],\quad Q_{13}=[Q_4,Q_8],\quad Q_{14}=[Q_5,Q_8],\quad Q_{15}=[Q_6,Q_8]. \tag{15b}$$

The O(4, 2) Lie algebra can then be constructed by taking appropriate linear combinations of Q_7 and Q_8. This reproduces the results of previous authors[5,7] with an important difference, namely, an explicit time dependence is present in (15b).[7]

Anderson, Kumei, and Wulfman[4] and Kumei[8] gave a detailed presentation of the methodology and results for a variety of systems including the free particle, harmonic oscillator, rigid rotator, symmetric top, and hydrogenlike atom.

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Response of a Gravitational-Wave Antenna to a Polarized Source

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The response of a gravitational wave antenna to linear, mixed, and randomly polarized sources is studied as a function of sidereal time, source coordinates, and antenna location and orientation. We find that the gravitational signals reported by Weber cannot be highly polarized tensor radiation coming from a single source at the Galactic nucleus.

Weber[1] has reported bursts of gravitational radiation apparently arriving from the direction of the Galactic center. The amount of power observed amounts to about $10^{-2}M_\odot c^2$/yr for a source at the Galactic center radiating isotropically. Experiment[2] shows that the spectral width is at least 10^2 Hz and probably more than 10^3 Hz. Estimating the efficiency of the detectors, one arrives at Galactic mass-loss rates around $10^4 M_\odot$/yr, which seems to be about 10^2 times larger than the upper limit allowed by astronomical data.[3] Although the range to the source has not been determined, we assume with Weber that the most probable location for a source is near the high-density Galactic nucleus.

Of the various reasonable assumptions made in estimating the Galactic mass-loss rate from the observed flux density, only the assumed isotropy of the source seems sufficiently assailable as to relinquish two orders of magnitude. Some of the variously proposed sources of gravitational radiation would not be expected to radiate isotropically and with random polarization. It is known that the degree of planar anisotropy in source motion is related to the degree of radiated linear polarization.[4] For example, the gravitational flux from a rotating pair of masses would be linearly polarized in the equatorial plane. More recently, Misner *et al.*[5] have considered the synchrotronlike modes of mass falling into a rotating super black hole. The resulting polarized gravitational radiation could fall into narrow angles about the galactic disk. In this Letter, we examine the consequences of a polarized source for Weber's antennas and others. We also study the effects of changing the antenna orientation and polarization angle.

The driving signal for any type of gravitational tensor wave detector is proportional to certain components of the local Riemann curvature tensor given by the vacuum solutions of Einstein's field equations, and is a scalar of the form[6] $-c^2R_{0\alpha 0}{}^{\mu} l^\alpha l_\mu$. Here, l^α is a vector along the antenna axis. The general response of a gravita-